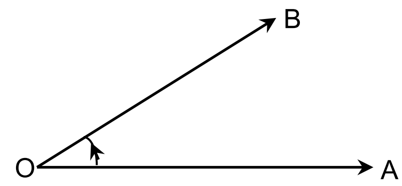
TRIGONOMETRIC FUNCTIONS

Definitions

Angle : An angle measures the amount of rotation of a revolving line with respect to a fixed line.
If the rotation is in clock-wise sense, the angle measured is said to be negative and positive if the rotation is in anti-clockwise sense.

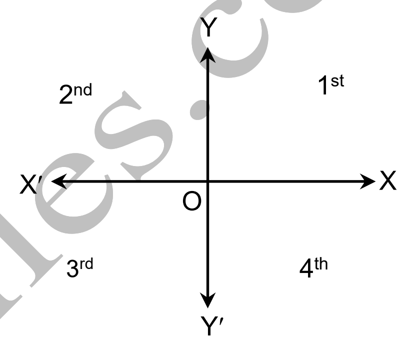


Right Angle : The measure of the angle is a right angle if the revolving line starting from its initial position to final position describes one quarter of a circle.

Angle in Standard Position : An angle is said to be in standard position if the vertex of the angle is at the origin O and its initial side is along the positive direction of x –axis.

Quadrants

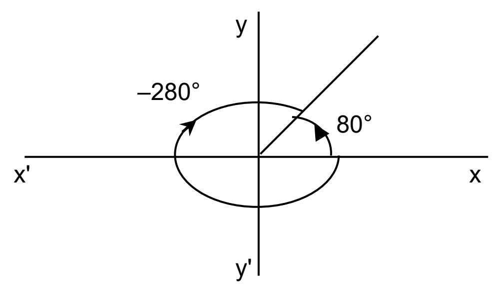
Let XOX′ and YOY′ be two mutually perpendicular lines in a plane and OX be the initial half line. The whole plane is divided into 4 different regions XOY, YOX′ , X′OY′ and Y′OX. These regions are called quadrants and are respectively called 1st, 2nd, 3rd and 4th quadrants according as the terminal side lies in the 1st, 2nd , 3rd and 4th quadrants. If the terminal side coincides with one of the axes, then angle is called a quadrant angle.



Angle in a Particular Quadrant : An angle is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant and initial side is along the +ve direction of x-axis.

Coterminal Angles

Two angles with different measures but having the same initial sides & the same terminal sides are known as co-terminal angles. For example:- The angles with measure 80^o and -280^o are co-terminal angles.



Various Types of Angles

An angle θ is said to be:

- an acute angle, if $0^0 \le \theta < 90^0$
- an obtuse angle, if $90^0 < \theta < 180^0$
- a reflex angle, if $180^0 < \theta < 360^0$
- a right angle, if $\theta = 90^0$
- a straight angle, if $\theta = 180^0$

Measurement of Angles

(a) **Sexagesimal System (English System)** : In this system, the right angle is divided into 90 equal parts and each part is called a **degree**. Each degree is further sub divided into 60 equal subparts, each subpart is called a **minute** and each minute is again sub-divided into 60 equal subparts, each subpart is called a **second**. Thus we have

1 rt. angle = 90 degrees, written as 90^o
1 degree or 1^o = 60 minutes, written as $60'$
1 minute or $1'$ = 60 seconds, written as $60''$

Hence an angle containing 79 degrees, 24 minutes and 9 seconds is generally written as $79^o\ 24'\ 9''$. One advantage of taking straight angle as 180^o is that it is easily subdivided into halves, thirds, fourths etc; something that is not possible if we take the straight angle to be 100 or 150 or 200 units.

(b) **Circular System :** This system of measurement of an angle is used in all the higher branches of mathematics. In this system the unit for the measurement of an angle is a radian which is obtained as follows : Take any circle whose centre is O and cut off an *arc* AB *equal in length to the radius of the circle*. Join OA and OB. Then $\angle AOB$ is a radian.

Illustration 1

Find the number of minutes in 4.5 degrees.

4.5 degrees = 4.5 × 60 minutes ($\because$ 1° = 60′)
= 270 minutes

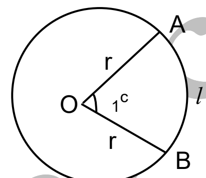
Illustration 2

Find the number of seconds in 4.5 degrees.

4.5 degrees = 270 minutes = 270 × 60 seconds ($\because$ 1′ = 60′)
= 16, 200 seconds

Definitions

A radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. 1 radian is written as 1^c or 1.



Result

If θ is the number of the radians in an angle subtended at the centre of a circle of radius r by an arc of length l, then $\theta = \frac{l}{r}$.

Illustration 3

Find the length of the arc of a circle of radius 5 cm subtending a central angle measuring 15°.

Let s be the length of the arc subtending an angle θ at the center of a circle of radius r, $\theta = \frac{\text{arc}}{\text{radius}} = \frac{s}{r}$

Here, r = 5cm and $\theta = 15^0 = \left(15 \times \frac{\pi}{180^0}\right)^c$

Now, $\theta = \left(\frac{\pi}{12}\right)^c$

Using, $\theta = \frac{s}{r}$ we get, s = 5π/12 cm

Note :

θ should always be expressed in radians and l, r must be taken in the same units and radian is a constant angle i.e. its measure does not depend on the size of the circle that we take.

Relation between Degrees and Radians.

We know that π radians = 180°

⇒ 1 radian = $\left(\frac{180}{\pi}\right)^o$ ⇒ 1 rad. = (180 × 0.31831)° $\left(\because \frac{1}{\pi} = .31831\right)$

1 rad. = 57° 17′ 46 ″ nearly

Again, 180° = π radians

⇒ $1^o = \frac{\pi}{180}$ radians.

⇒ $1^o = \frac{3.1416}{180}$ radians

⇒ 1° = .01745 radians nearly.

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Illustration 4

Express 45° 20′ 10″ in radian measure (π = 3.1416)

$$\left(\frac{16321}{360}\right)^o = \frac{16321}{360}\times\frac{\pi}{180}\text{ radians}$$

$$= \frac{16321}{360}\times\frac{3.1416}{180} = .79\text{ radians}$$

Illustration 5

The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions per second. How fast is the train going ? (π = 3.14).

We have r = radius of the wheel = $\frac{4}{2}$ ft. = 2 ft.

∴ circumference of the wheel = 2πr = 2 × 3.14 × 2 = 12.56 ft.

Since the wheel makes 6 revolutions per second,

∴ distance traversed in 1 second = 6 × 12.56 ft = 75.36 ft.

Hence the required velocity of the train

= 75.36 ft. / sec.

Illustration 6

A horse trots uniformly along a circular track of radius 27 m. The angle subtended at the centre of the track by the arc passed over by the horse in 3 seconds is 70°. What distance will the horse pass over in $\frac{1}{2}$ minute?

Let s m be the distance passed over by the horse in 3 seconds.

Here r = radius of the circular track = 27 m

θ = angle subtended at the centre by arc of length s m

$$= 70° = 70 \times \frac{\pi}{180} = \frac{7\pi}{18}\text{ radians}$$

We have

$$s = r\,\theta = 27 \times \frac{7\pi}{18} = \frac{3}{2}\times 7\times\frac{22}{7} = 33\text{ m (approx)}.$$

Distance passed over by the horse per second = $\frac{33}{3} = 11\,\text{m}$

∴ The required distance that the horse will pass over in $\frac{1}{2}$ minute

= 11 × 30 m = 330 m

Illustration 7

The angles of a certain triangle are in A.P. If the ratio of the number of degree in the least angle to the number of radians in the greatest angle be as 60 : π, express the angles in degrees.

Let the three angles in A.P. be A – d, A, A + d which are measured in degrees.

Then the greatest angle = (A + d) degrees = (A + d) $\frac{\pi}{180}$ radians.

By the given condition, (A – d) : $\left\{(A+d)\frac{\pi}{180}\right\}$ = 60 : π

or, $\frac{A-d}{A+d}.\frac{180}{\pi} = \frac{60}{\pi}$, or, 3(A – d) = A + d or, d = $\frac{1}{2}$ A.

Also (A – d) + A + (A + d) = 180°

or, $A = 60°$

$\therefore\ d = \frac{1}{2}A = \frac{1}{2} \times 60° = 30°.$

$\therefore\ A - d = 60° - 30° = 30°,\ A = 60°$ and $A + d = 60° + 30° = 90°$

Hence the required angles in degrees are 30°, 60° and 90°.

Illustration 8

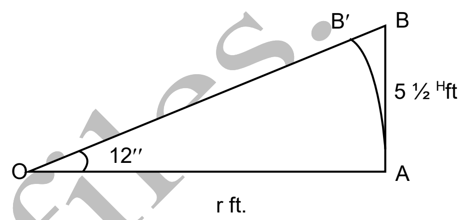
At what distance does a man of height $5\frac{1}{2}$ ft. subtend an angle 12″? (Given $\pi = \frac{22}{7}$)

Let AB be the man of height $5\frac{1}{2}$ ft. and let AB subtend an angle of 12″ at the point O.

Let OA = r ft. We have

$$\theta = 12'' = \frac{12}{60 \times 60} \times \frac{\pi}{180} \text{ radians}$$

Since the angle 12″ = ∠AOB is very small, AB is also very small compared to the distance OA. So if we draw an arc AB′ of a circle with O as centre and A = r ft. as radius such that arc AB′ subtends an angle 12″ at O, then arc AB′ = AB (nearly),



or, $s = 5\frac{1}{2}$, or, $r\theta = 5\frac{1}{2}$,

or, $r \times \frac{12}{60 \times 60} \times \frac{\pi}{180} = \frac{11}{2}$,

or, $r = \frac{11}{2} \times \frac{60 \times 60 \times 180 \times 7}{12 \times 22}$ ft. $[\because \pi = 22/7]$

$= 94500$ ft.

Relation between side and interior angles of a regular polygon

Let us consider a polygon of n equal sides

Then, sum of interior angles = $(2n - 4) \times 90^o$

If the polygon is regular each interior angle of the polygon = $\frac{2n-4}{n} \times 90^o$

Each exterior angle of a regular polygon = $\frac{360^o}{n}$

Illustration 9

The number of sides of two regular polygons are as 5 : 4 and difference between their each interior angle is 9^0. Find the number of sides of two regular polygons.

Let the number of sides be 5n and 4n

According to the above condition,

$$\frac{(2 \times 5n - 4) \times 90^0}{5n} - \frac{(2 \times 4n - 4) \times 90^0}{4n} = 9^0$$ From here, n = 2

Hence, the number of sides are 10 and 8

Note: In a clock, angle traced by minute hand in one minute = 6°

Angle traced by hour hand in one minute = $\left(\frac{1}{2}\right)^o$

Therefore the minute hand leads the hour hand by $5\frac{1}{2}^o$ every minute.

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Illustration 10

Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.

Angle traced by hour hand in 12 hours = 360°

Angle traced by hour hand in 7 hrs 20 minutes = $\left(\frac{360}{12}\times 7\right)^{o} + 10^{o} = 220^{o}$

Also, the angle traced by minute hand in 60 min = 360°

The angle traced by minute hand in 20 min = $\left(\frac{360}{60}\times 20\right)^{0} = 120^{0}$

Hence, the required angle between two hands is 220° – 120° = 100°

Practice Assignment -I

1. Find the degree measure corresponding to the following radian measures :

(i) $\frac{9\pi}{5}$ (ii) –4 (iii) $\frac{7\pi}{6}$ (iv) $\frac{11}{16}$ (v) $\frac{5\pi}{3}$

2. Find the radian measure corresponding to the following degree measures

(i) 300° (ii) 125° 30′ (iii) 7°30′ (iv) 25°

(v) –47°30 ′ (vi) 240° (vii) 520°

3. The angles of triangle are in the ratio 3 : 4 : 5. Find the smallest angle in degrees and greatest angle in radians.

4. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

5. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm. (Use $\pi = \frac{22}{7}$)

6. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.

7. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

8. Find the angle in radians through which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

9. The minute hand of the clock is 10 cm long. How far does the tip of the hand move in 20 minutes?

10. A wire 121 cm long is bent so as to lie along the arc of a circle of radius 180 cm. Find in degrees the angle subtended at the centre by the arc.

11. A man running along a circular track at the rate of 10 miles per hour traverses in 36 seconds, an arc which subtends an angle 56° at the centre, find the diameter of the circle.

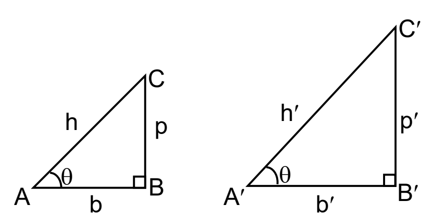
12. Find the distance from the eye at which a coin of 2 cm diameter should be held so as to conceal the full moon whose angular diameter is 31′.

13. The perimeter of a certain sector of a circle is equal to the length of the arc of the semicircle having the same radius. Express the angle of the sector in degrees, minutes and seconds.

14. The ratio of angle in one regular polygon is to that in another is 3 : 2 and the number of sides in the first is twice that in the second. Determine the number of sides of the two polygons.

Trigonometric Functions of Acute Angles

An angle whose measure is greater than 0° but less than 90° is called an acute angle. Consider a right angled triangle ABC with right angle at B. Side opposite to right angle is called the hypotenuse, side opposite to angle A is called perpendicular for angle A and side opposite to third angle is called base for angle A.



Any ratio of two sides of the triangle depends only on the measure of angle A that is on θ.

Definitions :

(i) $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}} = \dfrac{p}{h}$ (ii) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{b}{h}$

(iii) $\tan\theta = \dfrac{\text{perpendicular}}{\text{base}} = \dfrac{p}{b}$ (iv) $\cot\theta = \dfrac{\text{base}}{\text{perpendicular}} = \dfrac{b}{p}$

(v) $\sec\theta = \dfrac{\text{hypotenuse}}{\text{base}} = \dfrac{h}{b}$ (vi) $\text{cosec}\,\theta = \dfrac{\text{hypotenuse}}{\text{perpendicular}} = \dfrac{h}{p}$

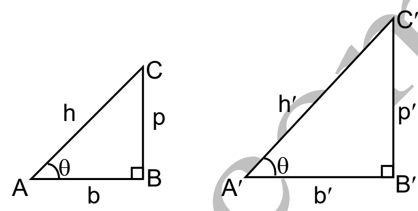
The abbreviations stand for sine, cosine, tangent, cotangent, secant, cosecant of θ respectively. These functions of angle θ are called trigonometric functions or trigonometric ratios.

Identity

An identity is a trigonometric equation, if it is true for all values of the angles in the domain of the trigonometric ratios involved.

For example. $\sin^2\theta + \cos^2\theta = 1, \forall\, \theta \in R$

$$\sec^2\theta = \tan^2\theta + 1, \forall\, \theta \in R - \{(2n+1)\frac{\pi}{2};\ n \in Z\}$$



A trigonometric equation is satisfied in general by only some points of the domain of the functions involved.

For e.g. $\sin\theta = 1$ is a trigonometric equation whose solution is $\theta = n\pi + (-1)^n\frac{\pi}{2}$.

Trigonometric Identities

(a) $\sin^2 A + \cos^2 A = 1$ or $\sin^2 A = 1 - \cos^2 A$ or $\cos^2 A = 1 - \sin^2 A$

(b) $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$ or $\sec A + \tan A = \dfrac{1}{\sec A - \tan A}$

where $A \neq n\pi + \dfrac{\pi}{2}, n \in Z$

(c) $1 + \cot^2 A = \text{cosec}^2 A$ or $\text{cosec}^2 A - \cot^2 A = 1$ or $\text{cosec}\, A + \cot A = \dfrac{1}{\text{cosec}\, A - \cot A}$

where $A \neq n\pi, n \in Z$

(d) $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$

(e) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A$

(f) $\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A$

(g) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$

Illustration 11

Prove $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

L.H.S. $= \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}\cdot\dfrac{1+\sin\theta}{1+\sin\theta}}$

$= \sqrt{\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\dfrac{(1+\sin\theta)^2}{\cos^2\theta}} = \dfrac{1+\sin\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$ = R.H.S

Illustration 12

Show that : $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cos^2 A)$

L.H.S $= \sin^8 A - \cos^8 A = (\sin^4 A)^2 - (\cos^4 A)^2$

$= (\sin^4 A - \cos^4 A)(\sin^4 A + \cos^4 A)$

$= (\sin^2 A - \cos^2 A)(\sin^2 A + \cos^2 A)[(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A]$

$= (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A\cos^2 A)$ [$\because \sin^2 A + \cos^2 A = 1$]

Illustration 13

Prove the following identity

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1+\sin\theta}{\cos\theta}$$

L.H.S. = $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$= \frac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1} = \frac{(\sec\theta + \tan\theta)[\tan\theta - \sec\theta + 1]}{\tan\theta - \sec\theta + 1}$

$= \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$

Illustration 14

If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

We have, $a\cos\theta - b\sin\theta = c$

⇒ $(a\cos\theta - b\sin\theta)^2 = c^2$ [Squaring both sides]

⇒ $a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta . \sin\theta = c^2$

⇒ $a^2(1 - \sin^2\theta) + b^2(1 - \cos^2\theta) - 2ab\cos\theta . \sin\theta = c^2$

⇒ $a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\cos\theta . \sin\theta = c^2$

⇒ $a^2\sin^2\theta + b^2\cos^2\theta + 2ab\cos\theta . \sin\theta = a^2 + b^2 - c^2$

⇒ $(a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

⇒ $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

Illustration 15

If $3\sin\theta + 5\cos\theta = 5$, show that $5\sin\theta - 3\cos\theta = \pm 3$

Given, $3\sin\theta + 5\cos\theta = 5$ ....(1)

Let, $5\sin\theta - 3\cos\theta = x$ ....(2)

Squaring (1) and (2) and adding, we get

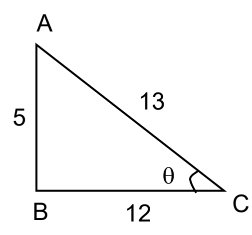
$(9\sin^2\theta + 25\cos^2\theta + 30\sin\theta\cos\theta) + (25\sin^2\theta + 9\cos^2\theta - 30\sin\theta . \cos\theta) = 25 + x^2$

⇒ $9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$

⇒ $34 = 25 + x^2 \Rightarrow x^2 = 9$, ∴ $x = \pm 3$

Illustration 16

If θ is an acute angle and $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$ and $\sec\theta$.



Given, $\sec\theta + \tan\theta = \frac{3}{2}$ ....(1)

We have, $\sec^2\theta - \tan^2\theta = 1$

⇒ $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

⇒ $\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta} = \frac{2}{3}$ .....(2)

Adding (1) and (2), we get

$2\sec\theta = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$

∴ $\sec\theta = \frac{13}{12}$ $\tan\theta = \frac{5}{12}$ and $\sin\theta = \frac{5}{13}$

Illustration 17

If cosec θ – sin θ = m and sec θ – cos θ = n, eliminate θ.

Given cosec θ – sin θ = m or $\frac{1}{\sin\theta} - \sin\theta = m$

or $\frac{1-\sin^2\theta}{\sin\theta} = m$ or $\frac{\cos^2\theta}{\sin\theta} = m$ [∵ $1 = \sin^2\theta + \cos^2\theta$] ....(1)

Again, sec θ – cos θ = n or $\frac{1}{\cos\theta} - \cos\theta = n$

or $\frac{1-\cos^2\theta}{\cos\theta} = n$ or $\frac{\sin^2\theta}{\cos\theta} = n$ .....(2)

From (1), $\sin\theta = \frac{\cos^2\theta}{m}$ .....(3)

Putting in (2), we get $\frac{\cos^4\theta}{m^2\cos\theta} = n$ or $\cos^3\theta = m^2 n$

∴ $\cos\theta = (m^2n)^{1/3}$ or $\cos^2\theta = (m^2n)^{2/3}$ .....(4)

From (3), $\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2n)^{2/3}}{m} = \frac{m^{4/3}n^{2/3}}{m} = m^{1/3}n^{2/3} = (mn^2)^{1/3}$

∴ $\sin^2\theta = (mn^2)^{2/3}$ .....(5)

Adding (4) and (5), we get

$(m^2n)^{2/3} + (mn^2)^{2/3} = \cos^2\theta + \sin^2\theta$ or $(m^2n)^{2/3} + (mn^2)^{2/3} = 1.$

Illustration 18

Eliminate θ between the equation

a sec θ + b tan θ + c = 0 and p sec θ + q tan θ + r = 0.

Given, a sec θ + b tan θ + c = 0 ...(1)

and p sec θ + q tan θ + r = 0. ....(2)

Solving (1) and (2) by cross multiplication method, we have

$$\frac{\sec\theta}{br-qc} = \frac{\tan\theta}{pc-ar} = \frac{1}{aq-pb}$$

(i) (ii) (iii)

From (i) and (iii), we get $\sec\theta = \frac{br-qc}{aq-pb}$ ....(3)

From (ii) and (iii), we get $\tan\theta = \frac{pc-ar}{aq-pb}$ ...(4)

∵ $\sec^2\theta - \tan^2\theta = 1$

∴ $\left(\frac{br-qc}{aq-pb}\right)^2 - \left(\frac{pc-ar}{aq-pb}\right)^2 = 1$

or $(br-qc)^2 - (pc-ar)^2 = (aq-pb)^2$

Illustration 19

If 2y cos θ = x sin θ and 2x sec θ – y cosec θ = 3, show that $x^2 + 4y^2 = 4$.

We have 2y cos θ = x sin θ

or, $\frac{\cos\theta}{x} = \frac{\sin\theta}{2y} = k$ (say)

Then cos θ = kx and sin θ = 2ky

Again $2x \sec\theta - y \operatorname{cosec}\theta = 3$.

or, $\frac{2x}{\cos\theta} - \frac{y}{\sin\theta} = 3$, or, $\frac{2x}{kx} - \frac{y}{2ky} = 3$, or, $\frac{2}{k} - \frac{1}{2k} = 3$,

or, $\frac{4-1}{2k} = 3$, or, $\frac{3}{2k} = 3$, or, $2k = 1$,

or, $k = \frac{1}{2}$, $\therefore$ $\cos\theta = \frac{x}{2}$ and $\sin\theta = 2 \cdot \frac{1}{2} \cdot y = y$

$\therefore$ $\cos^2\theta + \sin^2\theta = \frac{x^2}{4} + y^2 = 1$ or, $x^2 + 4y^2 = 4$.

Illustration 20

If $\tan\theta = \frac{a}{b}$, find $\frac{\sin\theta}{\cos^8\theta} + \frac{\cos\theta}{\sin^8\theta}$

$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{a^2}{b^2} = \frac{a^2+b^2}{b^2}$; $\therefore$ $\cos^2\theta = \frac{b^2}{a^2+b^2}$.

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2}$; $\therefore$ $\sin^2\theta = \frac{a^2}{a^2+b^2}$

$\therefore \sin^8\theta = \left(\frac{a^2}{a^2+b^2}\right)^4 = \frac{a^8}{(a^2+b^2)^4}$, $\cos^8\theta = \frac{b^8}{(a^2+b^2)^4}$

and $\sin\theta = \pm\frac{a}{\sqrt{a^2+b^2}}$, $\cos\theta = \pm\frac{b}{\sqrt{a^2+b^2}}$.

Given expression $= \pm\frac{a}{\sqrt{a^2+b^2}} \cdot \frac{(a^2+b^2)^4}{b^8} \pm \frac{b}{\sqrt{a^2+b^2}} \cdot \frac{(a^2+b^2)^4}{a^8} = \pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left(\frac{a}{b^8} + \frac{b}{a^8}\right)$

Illustration 21

Eliminate θ from the equations : $\tan\theta - \cot\theta = a$ and $\sin\theta + \cos\theta = b$.

We have

$a^2 = \tan^2\theta + \cot^2\theta - 2\tan\theta \cdot \cot\theta = (1 + \tan^2\theta) + (1 + \cot^2\theta) - 4$

or, $a^2 + 4 = \sec^2\theta + \operatorname{cosec}^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta \cdot \sin^2\theta}$

and $b^2 = (\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta \cdot \cos\theta = 1 + 2\sin\theta \cdot \cos\theta$

or, $(b^2 - 1)^2 = (2\sin\theta\cos\theta)^2 = 4\sin^2\theta \cdot \cos^2\theta$

$\therefore$ $(a^2 + 4)(b^2 - 1)^2 = \frac{1}{\cos^2\theta \cdot \sin^2\theta} \times 4\sin^2\theta \cdot \cos^2\theta = 4$

i.e. $(a^2 + 4)(b^2 - 1)^2 = 4$, which is the required result.

Illustration 22

Eliminate θ from the equation

$x\sin\theta - y\cos\theta = \sqrt{x^2+y^2}$ and $\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2}$.

We have

$x\sin\theta - y\cos\theta = \sqrt{x^2+y^2}$ ....(1)

and $\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{x^2+y^2}$ ....(2)

Squaring both sides of (1), we get

$x^2 \sin^2\theta + y^2 \cos^2\theta - 2xy \sin\theta \cos\theta = x^2 + y^2$

or, $x^2 (1 - \cos^2\theta) + y^2 (1 - \sin^2\theta) - 2xy \sin\theta \cos\theta = x^2 + y^2$

or, $x^2 \cos^2\theta + y^2 \sin^2\theta + 2xy \sin\theta \cos\theta = 0$

or, $(x \cos\theta + y \sin\theta)^2 = 0$, or, $x \cos\theta + y \sin\theta = 0$

or, $x \cos\theta = -y \sin\theta$,

or, $\frac{\cos\theta}{-y}=\frac{\sin\theta}{x}=\frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\sqrt{y^2+x^2}}=\frac{1}{\sqrt{x^2+y^2}}$

$\therefore$ $\cos\theta = -\frac{y}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$

Substituting these valued of $\cos\theta$ and $\sin\theta$ in (2), we get

$\frac{1}{a^2}\cdot\frac{y^2}{x^2+y^2}+\frac{1}{b^2}\cdot\frac{x^2}{x^2+y^2}=\frac{1}{x^2+y^2}$

or, $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$, or, $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, which is the required eliminant.

Practice Assignment -II

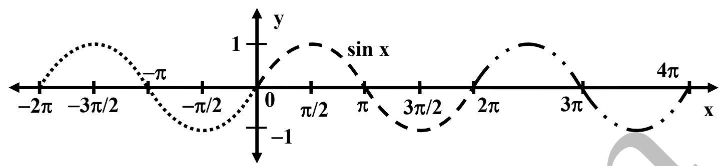
Prove the following identities

1. $(1 + \tan\alpha \tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha\ \sec^2\beta$.
2. $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ = sec A cosec A + 1.
3. $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$.
4. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$.
5. $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-1}+\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta+1}=2\sec^2\theta$.
6. (secA – cosecA) (1 + tanA + cot A) = tan A secA – cot A cosec A.
7. $(\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2 = 2 \tan A \cot B (\operatorname{cosec} A + \sec B)$.
8. $\frac{2\sin\theta\tan\theta(1-\tan\theta)+2\sin\theta\sec^2\theta}{(1+\tan\theta)^2}=\frac{2\sin\theta}{1+\tan\theta}$.
9. $\frac{\cot^2\theta(\sec\theta-1)}{1+\sin\theta}=\sec^2\theta.\frac{1-\sin\theta}{1+\sec\theta}$
10. $(\operatorname{cosec}\theta - \sec\theta)(\cot\theta - \tan\theta) = (\operatorname{cosec}\theta + \sec\theta)(\sec\theta \operatorname{cosec}\theta - 2)$
11. If $\sin x + \sin^2 x = 1$, then prove that $\cos^2 x + \cos^4 x = 1$.
12. If $\sec\theta + \tan\theta = 4$, find $\sec\theta$ and $\tan\theta$.
13. If $\sin\alpha + \operatorname{cosec}\alpha = 2$, prove that $\sin^n\alpha + \operatorname{cosec}^n\alpha = 2$
14. If (1 – sinA)(1 – sin B)(1 – sinC) = (1 + sinA)(1 + sinB)(1 + sinC), prove that each side is equal to ± cos A cos B cos C
15. If cosx + sinx = $\sqrt{2}$ cosx, prove that cosx – sinx = $\sqrt{2}$ sinx
16. If $\sin\theta$ and $\cos\theta$ are the roots of $ax^2 - bx + c = 0$ show that $a^2 - b^2 + 2ac = 0$
17. If $\cot\theta(1 + \sin\theta) = 4m$ and $\cot\theta(1 - \sin\theta) = 4n$, then prove that $(m^2 - n^2)^2 = mn$
18. If $\frac{\cos\alpha}{\cos\beta}=a,\frac{\sin\alpha}{\sin\beta}=b$, then prove that $(a^2 - b^2)\sin^2\beta = a^2 - 1$
19. If $\operatorname{cosec}\theta - \sin\theta = a$, $\sec\theta - \cos\theta = b$, then prove that $a^2b^2(a^2 + b^2 + 3) = 1$

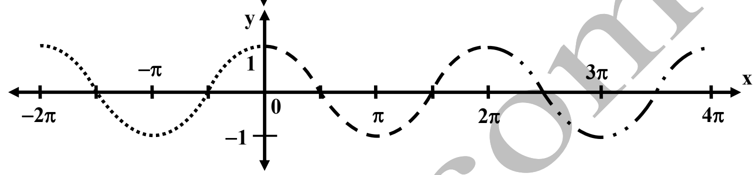
20. If $c\cos^3\theta + 3c\cos\theta\sin^2\theta = m$, $c\sin^3\theta + 3c\cos^2\theta\sin\theta = n$ then prove that $(m+n)^{2/3} + (m-n)^{2/3} = 2c^{2/3}$.

Graphs of Trigonometric Functions

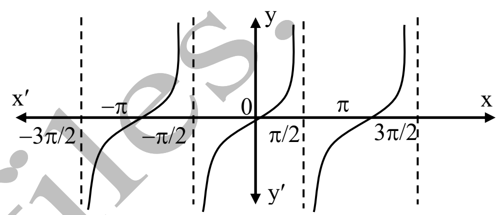
Sine function: The function that associates each real number x to sin x is called the sine function. Here x is the radian measure of the angle. The domain of the sine function is R and range is [–1, 1]



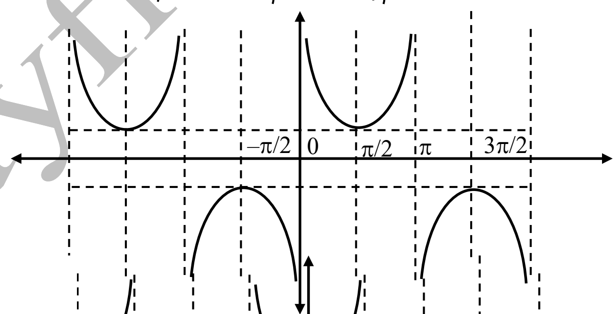
Consine function: The function that associates each real number x to cos x is called the cosine function. Here x is the radian measure of the angle.
The domain of the cosine function is R and the range is [–1 , 1]



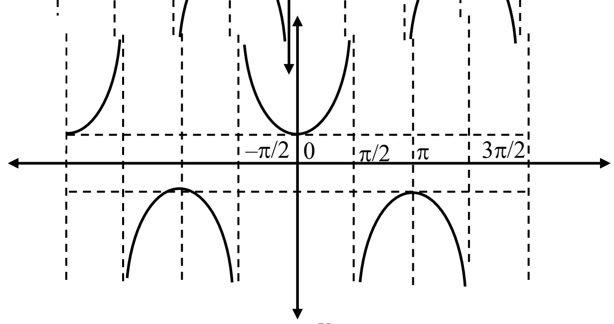
Tangent function: The function that associates a real number x to tan x is called the tangent function. Clearly, the tangent function is not defined at odd multiples of $\pi/2$ i.e. $\pm\pi/2$ $\pm 3\pi/2$ etc. So, the domain of the tangent function is $R - \{(2n+1)\pi/2 \mid n \in Z\}$. Since it takes every value between $-\infty$ and ∞ So, the range is R



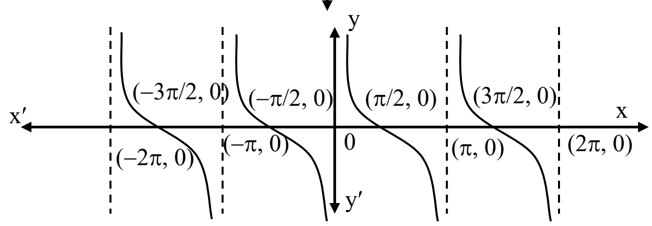
Cosecant function: The function that associates a real number x to cosec x is called the cosecant function.
Clearly, cosec x is not defined at $x = n\pi$, $n \in Z$ i.e. 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$ etc. So, its domain is $R - \{n\pi \mid n \in Z\}$. Since cosec $x \geq 1$ or cosec $x \leq -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$.



Secant function: The function that associates a real number x to sec x is called the secant function.
Clearly, sec x is not defined at odd multiples of $\pi/2$ i.e. $(2n+1)\pi/2$, where $n \in Z$. So, its domain is $R - \{(2n+1)\pi/2 \mid n \in Z\}$. Also, $|\sec x| \geq 1$, therefore its range is
$(-\infty, -1] \cup [1, \infty)$.



Cotangent function: The function that associates a real number x to cot x is called the cotangent function.
Clearly, cot x is not defined at $x = n\pi$, $n \in Z$. i.e. at $n = 0, \pm\pi, \pm 2\pi$, etc. So domain of cot x is $R - \{n\pi \mid n \in Z\}$.
The range of f(x) = cot x is R.



Behaviour of cos θ and sin θ as θ varies from 0 to 2π

In I Quadrant :
- cos θ decreases from 1 to 0; & is + ve
- sin θ increases from 0 to 1. & is + ve
- tan θ increases from 0 to ∞ & is + ve
- cot θ decreases from ∞ to 0 & is + ve
- sec θ increases from 1 to ∞ & is + ve
- cosec θ decreases from ∞ to 1 & is + ve

In II Quadrant :

- cos θ decreases from 0 to -1; & is $-$ve
- sin θ decreases from 1 to 0. & is $+$ve
- tan θ increases from $-\infty$ to 0 & is $-$ ve
- cot θ decreases from 0 to $-\infty$ & is $-$ve
- sec θ increases from $-\infty$ to -1 & is $-$ve
- cosec θ increases from 1 to ∞ & is $+$ve

In III Quadrant :

- cos θ increases from -1 to 0; & is $-$ve
- sin θ decreases from 0 to -1 & is $-$ve
- tan θ increases from 0 to $+\infty$ & is $+$ve
- cot θ decreases from $+\infty$ to 0 & is $+$ve
- sec θ decreases from -1 to $-\infty$ & is $-$ve
- cosec θ increases from $-\infty$ to -1 & is $-$ve

In IV Quadrant :

- cos θ increases from 0 to 1; & is $+$ve
- sin θ increases from -1 to 0 & is $-$ve
- tan θ increases from $-\infty$ to 0 & is $-$ve
- cot θ decreases from 0 to $-\infty$ & is $-$ve
- sec θ decreases from $+\infty$ to 1 & is $+$ve
- cosec θ decreases from -1 to $-\infty$ & is $-$ve

Domain and Range of Trigonometric Functions

	Domain	Range
Sin A	R	$[-1,1]$
Cos A	R	$[-1, 1]$
Tan A	$R - \{ (2n + 1)\pi/2 \mid n \in Z\}$	$(-\infty, \infty) = R$
Cosec A	$R - \{ n\pi \mid n \in Z\}$	$(-\infty, -1] \cup [1,\infty)$
Sec A	$R - \{(2n+1)\pi/2 \mid n \in Z\}$	$(-\infty, -1] \cup [1,\infty)$
Cot A	$R - \{n\pi \mid n \in Z\}$	$(-\infty, \infty) = R.$

Thus, $|\sin A| \le 1$, $|\cos A| \le 1$, $|\sec A| \ge 1$, $|\text{cosec } A| \ge 1$ for all values of A for which the functions are defined.

Periodic Functions

A function f(x) is said to be periodic if $f(x + p) = f(x)$, where $p \neq 0$ is the least positive real number called period of function.

For Example : cos θ, sin θ are the periodic functions, each of period 2π & tan θ is the periodic function of period π.

Some important results on periodic functions

(i) If f(x) is a periodic function with period T and $a, b \in R$ such that $a > 0$ then $f(ax + b)$ is periodic with period $T/|a|$.
For example sin x is periodic with period 2π Therefore $\sin(3x + 2)$ is periodic with $2\pi/3$

(ii) If $f_1(x), f_2(x), f_3(x)$ are periodic functions with periods T_1, T_2 and T_3 respectively then $a_1f_1(x) + a_2f_2(x) + a_3f_3(x)$ is a periodic function with period equal to LCM of T_1, T_2 and T_3 where a_1, a_2 and a_3 are real numbers

Note : If k be any integer, then

$\cos(2k\pi + \theta) = \cos\theta$

$\sin(2k\pi + \theta) = \sin\theta$

$\tan(k\pi + \theta) = \tan\theta$

T-Ratio's of allied angles

Trigon-ometric	$-\theta$	$90^0-\theta$ or	$90^0+\theta$ or	$180^0-\theta$ or	$180^0+\theta$ or	$270^0-\theta$ or	$270^0+\theta$ or	$360^0-\theta$ or

Ratio		$\left(\frac{\pi}{2}-\theta\right)$	$\left(\frac{\pi}{2}+\theta\right)$	$\pi-\theta$	$\pi+\theta$	$\left(\frac{3\pi}{2}-\theta\right)$	$\left(\frac{3\pi}{2}+\theta\right)$	$2\pi-\theta$
sin	– sin θ	cos θ	cos θ	sin θ	–sin θ	–cos θ	–cos θ	–sin θ
cos	cos θ	sin θ	–sin θ	–cos θ	–cos θ	– sin θ	sin θ	cos θ
tan	– tan θ	cot θ	– cot θ	–tan θ	tan θ	cot θ	–cot θ	–tan θ

Rule: T-function ($n \times 90^o \pm \theta$)

If n is odd , sin θ ⇌ cos θ

tan θ ⇌ cot θ

sec θ ⇌ cosec θ

if n is even then function will remain same and sign convention can be taken from the four quadrants.

For all values of θ,

cos (– θ) = cos θ

sin (– θ) = –sin θ

tan (– θ) = – tan θ

(1) Clearly in first quadrant sin θ, cos θ, tan θ, cot θ, sec θ and cosec θ are all positive as x, y are positive.

(2) In second quadrant x is negative and y is positive, therefore, only sin θ and cosec θ are positive.

(3) In third quadrant, x and y are both negative, therefore, only tan θ and cot θ are positive.

(4) In fourth quadrant, x is positive and y is negative, therefore, only cos θ and sec θ are positive.

Quadrant →	**I**	**II**	**III**	**IV**
sin θ	+	+	–	–
cos θ	+	–	–	+
tan θ	+	–	+	–
cosec θ	+	+	–	–
sec θ	+	–	–	+
cot θ	+	–	+	–

Even and odd functions

A function f(x) is said to be even or odd according as f(– x) = f(x) & f(–x) = –f(x) ∀ x ∈ R∈ domain of the function.
For example cos θ and sec θ are even functions but remaining t-ratios are odd functions

Value of Trigonometric Ratios of Standard Angles

Angle θ → T. Ratios ↓	**0°**	**30°**	**45°**	**60°**	**90°**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cosec θ	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cot θ	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Illustration 23

Prove that $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is possible for x = y, where $x \neq 0, y \neq 0$

We know that $\sec^2\theta \geq 1$, $\frac{4xy}{(x+y)^2} \geq 1$

$\Rightarrow \quad (x + y)^2 - 4xy \leq 0$

Hence, the given equation is possible if $\frac{4xy}{(x+y)^2} = 1$ or x = y

Illustration 24

Find period of $\sin\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{\pi x}{2}\right)$

Period of $\sin\left(\frac{2\pi x}{3}\right)$ is 3 and period of $\cos\frac{\pi x}{2}$ is 4

As LCM of 3 and 4 is 12, hence period is 12.

Note : This is not a universal rule. For example if f(x) = |sin x| + |cos x|. The period of |sinx| is π and the period of |cos x| is $\pi$and if we take L.C.M then we are expecting its period is π, but actually its period is $\pi/2$. To remove this confusion if $\phi(x)$ is of the form f (sin x) + f (cos x), f (tan x) + f(cot x), f (sec x) + f (cosec x) and f is an even function then we have to divide a period by 2.This can also apply if they are in product form.

Illustration 25

Which trigonometric ratios are positive for the following angles:

(i) 210° (ii) – 135° (iii) $\left(\frac{2\pi}{3}\right)^c$

(i) 210° is in the third quadrant hence only tan 210° and cot 210° are positive.

(ii) –135° lies in the third quadrant hence only tan (–135°) and cot (–135°) are positive

(iii) $\left(\frac{2\pi}{3}\right)^c = \left(\frac{2\pi}{3} \times \frac{180^\circ}{\pi}\right)^o$ = 120° which lies in the second quadrant.

Hence sin 120° and cosec 120° are positive.

Illustration 26

Find the values of the other five trigonometric functions in each of the following questions:

(i) $\tan\theta = \frac{5}{12}$, where θ is in third quadrant.

(ii) $\sin\theta = \frac{3}{5}$, where θ is in second quadrant.

(i) Since θ is in third quadrant,

$\therefore$ Only tan θ and cot θ are positive.

Now, $\tan\theta = \frac{5}{12}$. Therefore, $\cot\theta = \frac{12}{5}$, $\sin\theta = -\frac{5}{13}$, $\text{cosec}\theta = -\frac{13}{5}$, $\cos\theta = -\frac{12}{13}$ and $\sec\theta = -\frac{13}{12}$

(ii) Since θ is in the second quadrant,

$\therefore$ Only sin θ and cosec θ will be positive.

Now, $\sin\theta = \frac{3}{5}$. Therefore, $\text{cosec}\,\theta = \frac{5}{3}$, $\cos\theta = -\frac{4}{5}$, $\sec\theta = -\frac{5}{4}$, $\tan\theta = -\frac{3}{4}$ and $\cot\theta = -\frac{4}{3}$

Illustration 27

If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, find the value of $4\tan^2\theta - 3\,\text{cosec}^2\theta$.

Since $\pi < \theta < \frac{3\pi}{2}$, therefore $\tan\theta$ is positive while $\sin\theta$ is negative.

Now, $\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(-\frac{1}{2}\right)^2} = -\sqrt{1-\frac{1}{4}} = -\frac{\sqrt{3}}{2}$

$\Rightarrow$ $\text{cosec}\,\theta = -\frac{2}{\sqrt{3}}$

and, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$

$\therefore$ $4\tan^2\theta - 3\,\text{cosec}^2\theta = 4\times 3 - 3\times\frac{4}{3} = 8$

Illustration 28

Find the value of cosec (– 1410°)

$-1410° = -4\times 360° + 30°$

$\therefore$ $\text{cosec}(-1410°) = \text{cosec}(-4\times 360° + 30°) = \text{cosec}\,30° = 2$

Illustration 29

Find the value of $\tan\frac{19\pi}{3}$

$\frac{19\pi}{3} = 6\pi + \frac{\pi}{3} = 3.\,2\pi + \frac{\pi}{3}$

$\therefore$ $\frac{19\pi}{3} = \tan(3.\,2\pi + \frac{\pi}{3}) = \tan\frac{\pi}{3} = \sqrt{3}$

Illustration 30

If $\sec\theta = \sqrt{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find the value of $\frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta}$

If $\sec\theta = \sqrt{2}$ then $\cos\theta = \frac{1}{\sqrt{2}}$ (cos θ is +ve)

According to sign convention, $\sin\theta = -\sqrt{1-\cos^2\theta} = -\frac{1}{\sqrt{2}}$

$\tan\theta = -1$, then $\frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} = -1$

Illustration 31

Prove that $\cos 510^0 \cos 330^0 + \sin 390^0 \cos 120^0 = -1$

LHS = $\cos 510^0 \cos 330^0 + \sin 390^0 \cos 120^0$

$= \cos(360^0 + 150^0)\cos(360^0 - 30^0) + \sin(360^0 + 30^0)\cos(90^0 + 30^0)$

$= \cos 150^0 \cos 30^0 + \sin 30^0(-\sin 30^0) = -\frac{3}{4} - \frac{1}{4} = -1$

Illustration 32

Solve for x the equation : cosec (90° + A) + x cos A cot (90° + A) = sin (90° + A)

We have cosec (90° + A) + x cos A cot (90° + A) = sin (90° + A)

or, sec A + x cos A. (– tan A) = cos A,

or, $\frac{1}{\cos A} - x.\cos A.\frac{\sin A}{\cos A} = \cos A$, or, $\frac{1}{\cos A} - x \sin A = \cos A$

or, $x \sin A = \frac{1}{\cos A} - \cos A = \frac{1-\cos^2 A}{\cos A} = \frac{\sin^2 A}{\cos A}$ ∴ $x = \frac{\sin A}{\cos A} = \tan A$

Illustration 33

If, A, B, C be the angles of a triangle, prove that

$$\frac{\textbf{cosec (B + C) + cosec (C + A) + cosec (A + B)}}{\textbf{sec (270° + A) + sec(270° + B) + sec(270° + C)}} = 1$$

Since A, B, C are the angles of a triangle, we have

A + B + C = 180°,

∴ B + C = 180° – A, C + A = 180° – B and A + B = 180° – C

∴ cosec (B + C) = cosec (180° – A) = cosec A

Similarly, cosec (C + A) = cosec B and cosec (A + B) = cosec C.

Again sec (270° + A) = sec (3.90° + A) = cosec A.

Similarly, sec (270° + B) = cosec B, sec (270° + C) = cosec C.

$$\therefore \text{ L.H.S.} = \frac{\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C}{\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C} = 1.$$

Practice Assignment -III

1. Find the values of each of the following trigonometric ratios :

(i) sin 510° (ii) cosec (–405°) (iii) $\tan \frac{11\pi}{6}$

(iv) cos 855°. (v) sin 120° (vi) cos 150°

(vii) tan 135° (viii) cosec 330° (ix) sec(–300°)

(x) cot 1140° (xi) $\sec\left(\frac{-23\pi}{3}\right)$ (xii) $\operatorname{cosec}\left(\frac{35\pi}{6}\right)$

2. Find the value of $\frac{\sec 480^\circ . \operatorname{cosec} 570^\circ . \tan 330^\circ}{\sin 600^\circ \ \cos 660^\circ \ \cot 405^\circ}$

3. Find the value of $\sin^2(-300°) \cos^3(120°) + \cos^2(-240°) \sin^3 (390°)$

4. Verify that, cos 24° + cos 55° + cos 125° + cos 204° + cos 300° = $\frac{1}{2}$

5. Show that tan9°. tan27°. tan45°. tan63°. tan81° = 1.

6. Prove that $\sec\left(\frac{3\pi}{2} - \theta\right) \sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right) \tan\left(\theta - \frac{3\pi}{2}\right) = -1$.

7. Simplify

(i) $\frac{\cos(90^\circ + \theta)\sec(270^\circ + \theta)\sin(180^\circ - \theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ - \theta)\sin(180^\circ + \theta)}$

(ii) $\frac{\cos^2(2\pi - \theta)\tan^2(\pi + \theta)\sec^2(\pi - \theta)}{\sin^2(3\pi + \theta)\operatorname{cosec}^2(-\theta)\cot^2\left(\frac{\pi}{2} + \theta\right)}$

8. Prove that

(i) $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\dfrac{\pi}{2}+x\right)} = \cot^2 x$

(ii) $\cos\left(\dfrac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\dfrac{3\pi}{2}-x\right)+\cot(2\pi+x)\right] = 1$

9. In a ΔABC, prove that

(i) $\cos(A + B) + \cos C = 0$

(ii) $\tan\dfrac{A+B}{2} = \cot\dfrac{C}{2}$

(iii) $\dfrac{\sin(B+C)+\sin(C+A)+\sin(A+B)}{\sin(\pi+A)+\sin(3\pi+B)+\sin(5\pi+C)} = -1$

(iv) $\dfrac{\tan(B+C)+\tan(C+A)+\tan(A+B)}{\tan(\pi-A)+\tan(2\pi-B)+\tan(3\pi-C)} = 1$

10. If A, B, C, D are angles of a cyclic quadrilateral taken in order, then prove that

(i) $\cos A + \cos B + \cos C + \cos D = 0$.

(ii) $\sin(A + B) + \sin(C + D) = 0$

(iii) $\cos(B + C) = \cos(A + D)$

11. Find the values of other five trigonometric functions in each of the following problems :

(i) $\sin\theta = \dfrac{3}{5}$, θ in quadrant I

(ii) $\cos\theta = -\dfrac{1}{2}$, θ in quadrant II

(iii) $\cot\theta = \dfrac{3}{4}$, θ in quadrant III

(iv) $\sec\theta = \dfrac{13}{5}$, θ in quadrant IV

(v) $\tan\theta = -\dfrac{5}{12}$, θ in quadrant II

12. If $\sin\theta = \dfrac{3}{5}$, $\tan\phi = \dfrac{1}{2}$ and $\dfrac{\pi}{2} < \theta < \pi < \phi < \dfrac{3\pi}{2}$ find the value of $8\tan\theta - \sqrt{5}\sec\phi$.

13. If $\cos A = -\dfrac{24}{25}$ and $\cos B = \dfrac{3}{5}$ where $\pi < A < \dfrac{3\pi}{2}$ and $\dfrac{3\pi}{2} < B < 2\pi$, then find the value of $\sin A\cos B + \cos A\sin B$

14. Find the value of $\sin^2\dfrac{\pi}{18} + \sin^2\dfrac{\pi}{9} + \sin^2\dfrac{7\pi}{18} + \sin^2\dfrac{4\pi}{9}$.

15. Find the value of $\cos^2\dfrac{\pi}{16} + \cos^2\dfrac{3\pi}{16} + \cos^2\dfrac{5\pi}{16} + \cos^2\dfrac{7\pi}{16}$

16. Prove

(i) $\sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{4} = \dfrac{-1}{2}$

(ii) $2\sin^2\dfrac{\pi}{6} + \operatorname{cosec}^2\dfrac{7\pi}{6}\cos^2\dfrac{\pi}{3} = \dfrac{3}{2}$

(iii) $\cot^2\dfrac{\pi}{6} + \operatorname{cosec}\dfrac{5\pi}{6} + 3\tan^2\dfrac{\pi}{6} = 6$

(iv) $2\sin^2\dfrac{3\pi}{4} + 2\cos^2\dfrac{\pi}{4} + 2\sec^2\dfrac{\pi}{3} = 10$

(v) $3\sin\dfrac{\pi}{6}\sec\dfrac{\pi}{3} - 4\sin\dfrac{5\pi}{6}\cot\dfrac{\pi}{4} = 1$

17. If $\tan 25^\circ = a$, prove that $\dfrac{\tan 155^\circ - \tan 115^\circ}{1+\tan 155^\circ\tan 115^\circ} = \dfrac{1-a^2}{2a}$

18. Show that $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \ldots\ldots + \sin^2 90^\circ = 9\dfrac{1}{2}$

19. Evaluate $\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \dots\dots\dots + \log_{10}(\tan 89^\circ)$

20. Find x from the equation :

$x \cot(90^\circ + \theta) + \tan(90^\circ + \theta)\sin\theta + \text{cosec}(90^\circ + \theta) = 0$

Sum and Difference Formulae

i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ where $A \neq n\pi + \dfrac{\pi}{2}$, $B \neq n\pi + \dfrac{\pi}{2}$ and $A \pm B \neq m\pi + \dfrac{\pi}{2}$

vi) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$, $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$ where $A \neq n\pi, B \neq n\pi$ and $A \pm B \neq n\pi$

vii) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

vii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

Illustration 34

Find the value of (i) cos 15° (ii) sin 105° (iii) tan 105°.

(i) $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ.\cos 30^\circ + \sin 45^\circ.\sin 30^\circ$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\sin 105^\circ = \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ$

$$= \frac{1}{\sqrt{2}}.\frac{1}{2} + \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) $\tan 105^\circ = \tan(60^\circ + 45^\circ) = \dfrac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ.\tan 45^\circ}$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}.1} = \left(\frac{\sqrt{3}+1}{1-\sqrt{3}}\right)\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) = \frac{(\sqrt{3}+1)^2}{1-3} = \frac{3+1+2\sqrt{3}}{-2}$$

$$= \frac{4+2\sqrt{3}}{-2} = -(2+\sqrt{3})$$

Illustration 35

Show that tan 75° + cot 75° = 4.

$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad \dots(1)$$

and $\cot 75^\circ = \dfrac{1}{\tan 75^\circ} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ ....(2)

Now, L.H.S. = tan 75° + cot 75°

$= \dfrac{\sqrt{3}+1}{\sqrt{3}-1} + \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ [From (1) and (2)]

$= \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \dfrac{(4+2\sqrt{3})+(4-2\sqrt{3})}{3-1}$

$= \dfrac{8}{2}$ = 4 = R.H.S.

Illustration 36

If sin A = 3/5 and cos B = 9/41, 0 < A < π/2, 0 < B < π/2, find the values of the following :

(i) sin(A + B)

(ii) cos(A – B)

sin A = 3/5, cos A = 4/5

cos B = 9/41, sin B = 40/41

0 < A + B < π, in which sine function is positive

⇒ (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \dfrac{3}{5}\times\dfrac{9}{41}+\dfrac{4}{5}\times\dfrac{40}{41}=\dfrac{187}{205}$

–π/2 < A – B < π/2, in which cosine function is positive

⇒ (ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B = \dfrac{4}{5}\times\dfrac{9}{41}+\dfrac{3}{5}\times\dfrac{40}{41}=\dfrac{156}{205}$

Illustration 37

Prove that tan 70° = 2 tan 50° + tan 20°

We have $\tan 70^\circ = \tan(50^\circ + 20^\circ) = \dfrac{\tan 50^\circ + \tan 20^\circ}{1-\tan 50^\circ \tan 20^\circ}$

or, tan 70°(1 – tan 50° tan 20°) = tan 50° + tan 20°

or, tan 70° – tan 70° tan 50° tan 20° = tan 50° + tan 20°

or, tan 70° = tan 70° tan 50° tan 20° + tan 50° + tan 20°

= tan(90° – 20°) tan 50° tan 20° + tan 50° + tan 20°

= cot 20° tan 50° tan 20° + tan 50° + tan 20°

= tan 50° + tan 50° + tan 20° = 2 tan 50° + tan 20°

Illustration 38

Prove that cos 18° – sin 18° = $\sqrt{2}$ sin 27°

R.H.S = $\sqrt{2}$ sin 27° = $\sqrt{2}$ sin (45° – 18°)

= $\sqrt{2}$ (sin 45° cos 18° – cos 45° sin 18°)

$= \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos 18^\circ - \dfrac{1}{\sqrt{2}}\sin 18^\circ\right)$

= cos18° – sin 18° = L.H.S.

Illustration 39

Prove that $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

L.H.S. = $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x.\cos y + \cos x.\sin y}{\sin x.\cos y - \cos x.\sin y}$ ...(1)

Dividing the numerator and denominator of (1) by cos x cos y, we get

L.H.S. = $\frac{\tan x + \tan y}{\tan x - \tan y}$ = R.H.S.

Illustration 40

Prove that $\frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0$

First term of L.H.S. = $\frac{\sin(B-C)}{\cos B \cos C} = \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C}$

$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C}$ = tan B – tan C

similarly, second term of L.H.S. = tan C – tan A

and 3rd term of L.H.S. = tan A – tan B

Now L.H.S. = (tan B – tan C) + (tan C – tan A) + (tan A – tan B) = 0

Illustration 41

Prove that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

L.H.S. = $\cos^2 2x - \cos^2 6x$

$= 1 - \sin^2 2x - (1 - \sin^2 6x)$

$= \sin^2 6x - \sin^2 2x$

$= \sin^2 (6x + 2x) \sin (6x - 2x)$ [∵ $\sin (A + B) \sin (A + B) = \sin^2 A - \sin^2 B$]

$= \sin 8x \sin 4x = \sin 4x \sin 8x$

Illustration 42

If tan α = x + 1, tan β = x – 1, show that 2 cot (α – β) = x^2.

L.H.S. = 2 cot (α – β) = $\frac{2(\cot\alpha.\cot\beta + 1)}{\cot\beta - \cot\alpha}$

$= \frac{2\left(\frac{1}{1+x}.\frac{1}{x-1}+1\right)}{\left(\frac{1}{x-1}-\frac{1}{x+1}\right)}$

$= \frac{2\left(\frac{1}{x^2-1}+1\right)}{\frac{(x+1)-(x-1)}{x^2-1}} = 2.\frac{x^2}{2} = x^2$

Illustration 43

If 3 tan θ tan φ = 1 Prove that 2 cos (θ + φ) = cos (θ – φ)

Given, 3 tan θ tan φ = 1 or cot θ . cot φ = 3

Or $\frac{\cos\theta \cos\varphi}{\sin\theta \sin\varphi} = \frac{3}{1}$

By componendo and dividendo, we have

$$\frac{\cos\theta\cos\varphi+\sin\theta\sin\varphi}{\cos\theta\cos\varphi-\sin\theta\sin\varphi}=\frac{3+1}{3-1} \text{ or } \frac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)}=2$$

or $2\cos(\theta+\varphi)=\cos(\theta-\varphi)$

Illustration 44

Find the value of $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$, given that $\tan\alpha = 2\tan\beta$.

We have $\tan\alpha = 2\tan\beta$

or, $\frac{\sin\alpha}{\cos\alpha}=2.\frac{\sin\beta}{\cos\beta}$,

or, $\sin\alpha \,.\, \cos\beta = 2\cos\alpha \,.\, \sin\beta$

Now $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\sin\alpha.\cos\beta+\cos\alpha.\sin\beta}{\sin\alpha.\cos\beta-\cos\alpha.\sin\beta}=\frac{2\cos\alpha.\sin\beta+\cos\alpha.\sin\beta}{2\cos\alpha.\sin\beta-\cos\alpha.\sin\beta}$

$=\frac{3\cos\alpha\,.\sin\beta}{\cos\alpha.\,\sin\beta}=3.$

Illustration 45

If $\tan x - \tan y = a$ and $\cot y - \cot x = b$, prove that $1/a + 1/b = \cot(x - y)$.

We have

$\frac{1}{a}+\frac{1}{b}=\frac{1}{\tan x-\tan y}+\frac{1}{\cot y-\cot x}$

$=\frac{\cos x\,.\,\cos y}{\sin x\cos y-\cos x\sin y}+\frac{\sin x\sin y}{\sin x\cos y-\cos x\sin y}$

$=\frac{\cos x\cos y+\sin x\sin y}{\sin x\cos y-\cos x\sin y}=\frac{\cos(x-y)}{\sin(x-y)}$

$=\cot(x-y)$

Illustration 46

If $\tan\beta=\frac{\sin\alpha\cos\alpha}{2+\cos^2\alpha}$, prove that $3\tan(\alpha-\beta) = 2\tan\alpha$.

L.H.S. $= 3\tan(\alpha-\beta)$

$=3\,.\,\frac{\tan\alpha-\tan\beta}{1+\tan\alpha.\tan\beta}=3.\frac{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\alpha.\cos\alpha}{2+\cos^2\alpha}}{1+\frac{\sin\alpha}{\cos\alpha}.\frac{\sin\alpha.\cos\alpha}{2+\cos^2\alpha}}$

$=3\,.\,\frac{2\sin\alpha+\sin\alpha.\cos^2\alpha-\sin\alpha.\cos^2\alpha}{2\cos\alpha+\cos^3\alpha+\sin^2\alpha.\cos\alpha}$

$=3\,.\,\frac{2\sin\alpha}{2\cos\alpha+\cos\alpha(\cos^2\alpha+\sin^2\alpha)}=\frac{6\sin\alpha}{2\cos\alpha+\cos\alpha}$

$=\frac{6\sin\alpha}{3\cos\alpha}=2\tan\alpha.$

Practice Assignment– IV

1. If $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ and $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$, find the values of
(i) $\sin(A + B)$ (ii) $\cos(A + B)$ (iii) $\tan(A + B)$
(iv) $\sin(A – B)$ (v) $\cos(A – B)$ (vi) $\tan(A – B)$

2. Find the value of (i) sin 75° (ii) tan 15° (iii) cos 15° (iv) cot 105°

3. Prove

(i) $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$

(ii) $\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$

(iii) $\sin(n + 1)x \sin(n + 2)x + \cos(n + 1)x \cos(n + 2)x = \cos x$

(iv) $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

4. Prove that

(i) $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

(ii) $\sqrt{3}(\tan 170° - \tan 140°) = 1 + \tan 170° \tan 140°$

(iii) $\tan 20° + \tan 25° + \tan 20° \tan 25° = 1$

(iv) $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

5. Prove that

(i) $\frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0} = \tan 54^0$ (ii) $\frac{\cos 8^o + \sin 8^o}{\cos 8^o - \sin 8^o} = \cot 37°$ (iii) $\frac{\cos 15^o + \sin 15^o}{\cos 15^o - \sin 15^o} = \sqrt{3}$

6. If $A + B = 45^O$, Show that $(1 + \tan A)(1 + \tan B) = 2$

7. If $A + B = \frac{5\pi}{4}$, prove that $\frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} = \frac{1}{2}$.

8. If $\tan\alpha = \sqrt{a\lambda}$, $\tan\beta = \sqrt{\frac{\lambda}{a}}$ and $\tan\gamma = \sqrt{\frac{\lambda}{a^3}}$, where $\lambda = 1 - a - a^2$, prove that $\alpha + \beta = \gamma$.

9. If $\cos(\alpha + \beta) = \frac{4}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, then prove that $\tan 2\alpha = \frac{56}{33}$.

10. If $\tan(A – B) = x$ and $\tan(A + B) = y$, prove that $\tan 2B = \frac{y - x}{1 + xy}$.

11. If $\tan\alpha = \frac{m}{m + 1}$ and $\tan\beta = \frac{1}{2m + 1}$, show that $\alpha + \beta = \frac{\pi}{4}$.

12. Prove that $\frac{\tan(A + B)}{\cot(A - B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$.

13. Prove that

(i) $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)\sin A$

(ii) $\sin^2\left(\frac{\pi}{12} + \frac{\theta}{2}\right) - \sin^2\left(\frac{\pi}{12} - \frac{\theta}{2}\right) = \frac{1}{2}\sin\theta$

(iii) $\cos^2\left(\frac{\pi}{6}-\frac{\theta}{2}\right)-\sin^2\left(\frac{\pi}{6}+\frac{\theta}{2}\right)=\frac{1}{2}\cos\theta$

(iv) $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

(v) $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

14. If $\tan B = \frac{2\sin A\sin C}{\sin(A+C)}$, prove that cot A, cotB, cot C are in AP.

15. If α and β are solutions of the equation $a\tan\theta + b\sec\theta = c$, then show that $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$.

16. If in a triangle ABC , sin A cos B = 1/4 , and 3 tan A = tan B, then prove that triangle is right angled.

17. Prove that $\frac{\tan^2 2\theta-\tan^2\theta}{1-\tan^2 2\theta\tan^2\theta}=\tan 3\theta\tan\theta$.

18. If $2\tan\beta + \cot\beta = \tan\alpha$, prove that $\cot\beta = 2\tan(\alpha-\beta)$.

Transformation Formulae

Sum and difference into products

(a) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

(b) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(c) $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

(d) $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

(e) $\left.\begin{aligned}\tan A+\tan B&=\frac{\sin(A+B)}{\cos A\cos B}\\ \tan A-\tan B&=\frac{\sin(A-B)}{\cos A\cos B}\end{aligned}\right]$, where $A, B \neq n\pi \pm \pi/2, n \in Z$

(f) $\left.\begin{aligned}\cot A+\cot B&=\frac{\sin(A+B)}{\sin A\sin B}\\ \cot A-\cot B&=\frac{\sin(B-A)}{\sin A\sin B}\end{aligned}\right]$, where $A, B \neq n\pi$, $n \in Z$

Product into sum or difference

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

T-Ratios of the sum of three angles

(a) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$
or $\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$

(b) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$
$\cos(A+B+C) = \cos A\cos B\cos C(1-\tan A\tan B - \tan B\tan C - \tan C\tan A)$

(c) $\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

Illustration 47

Express the sin 4θ + sin 3θ as product of sine and cosines.

sin 4θ + sin 3θ = $2\sin\left(\frac{4\theta+3\theta}{2}\right)\cos\left(\frac{4\theta-3\theta}{2}\right)$ $\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$ = $2\sin\frac{7\theta}{2}.\cos\frac{\theta}{2}$

Illustration 48

Prove that cos 18° – sin 18° = $\sqrt{2}$ sin 27°

L.H.S. = cos 18° – sin 18° = cos 18° – sin(90° – 72°)

$= \cos 18^\circ - \cos 72^\circ = 2\sin\frac{18^\circ+72^\circ}{2}\sin\frac{72^\circ-18^\circ}{2}$

$= 2\sin 45^\circ \sin 27^\circ = 2\frac{1}{\sqrt{2}}\sin 27^\circ = \sqrt{2}\sin 27^\circ$

Illustration 49

Prove that $\dfrac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)} = \tan\theta$

We have,

$$\text{L.H.S.} = \frac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}$$

$$= \frac{\sin(\theta+\phi)+\sin(\theta-\phi)-2\sin\theta}{\cos(\theta+\phi)+\cos(\theta-\phi)-2\cos\theta}$$

$$= \frac{2\sin\theta\cos\phi-2\sin\theta}{2\cos\theta\cos\phi-2\cos\theta}$$

$$= \frac{2\sin\theta(\cos\phi-1)}{2\cos\theta(\cos\phi-1)} = \tan\theta = \text{R.H.S.}$$

Illustration 50

Prove that (cos α + cos β)² + (sin α + sin β)² = 4 cos²$\left(\frac{\alpha-\beta}{2}\right)$

$$\text{L.H.S.} = \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\cos^2\left(\frac{\alpha+\beta}{2}\right)+\sin^2\left(\frac{\alpha+\beta}{2}\right)\right] = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

Illustration 51

Prove that $\dfrac{\sin 5A + 2\sin 8A + \sin 11A}{\sin 8A + 2\sin 11A + \sin 14A} = \dfrac{\sin 8A}{\sin 11A}$

$$\text{L.H.S.} = \frac{(\sin 11A + \sin 5A) + 2\sin 8A}{(\sin 14A + \sin 8A) + 2\sin 11A}$$

$$= \frac{2.\sin\frac{11A+5A}{2}.\cos\frac{11A-5A}{2}+2\sin 8A}{2.\sin\frac{14A+8A}{2}.\cos\frac{14A-8A}{2}+2\sin 11A}$$

$$= \frac{2\sin 8A\cos 3A+2\sin 8A}{2\sin 11A\cos 3A+2\sin 11A} \qquad = \frac{2\sin 8A(\cos 3A+1)}{2\sin 11A(\cos 3A+1)} = \frac{\sin 8A}{\sin 11A}$$

Illustration 52

If sin x + sin y = p and cos x + cos y = q, show that cot $\frac{x-y}{2} = \pm\sqrt{\frac{p^2+q^2}{4-p^2-q^2}}$.

sin x + sin y = p and cos x + cos y = q

$\therefore \quad 2\sin\frac{x+y}{2}.\cos\frac{x-y}{2} = p$ .....(1) and

$2\cos\frac{x+y}{2}.\cos\frac{x-y}{2} = q$ .....(2)

Squaring (1) and (2), and then adding, we get

$= \quad 4\cos^2\frac{x-y}{2}\left(\sin^2\frac{x+y}{2}+\cos^2\frac{x+y}{2}\right) = p^2+q^2$

or, $\quad 4\cos^2\frac{x-y}{2} = p^2+q^2$, or, $\cos^2\frac{x-y}{2} = \frac{p^2+q^2}{4}$

$\therefore \quad \sin^2\frac{x-y}{2} = 1-\cos^2\frac{x-y}{2} = 1-\frac{p^2+q^2}{4} = \frac{4-p^2-q^2}{4}$

$\therefore \quad \cos^2\frac{x-y}{2} = \frac{\cos^2\frac{x-y}{2}}{\sin^2\frac{x-y}{2}} = \frac{p^2+q^2}{4-p^2-q^2}$

$\therefore \cot\frac{x-y}{2} = \pm\sqrt{\frac{p^2+q^2}{4-p^2-q^2}}$

Illustration 53

Express the 2 sin 3θ. cos θ as the sum or difference of cosines and sines.

2 sin 3θ. cos θ = sin(3θ + θ) + sin (3θ – θ) [∵ 2 sin A cos B = sin (A + B) + sin (A – B)]

= sin 4θ + sin 2θ

Illustration 54

Show that cos 10° + cos110° + cos 130° = 0.

L.H.S. = (cos 110° + cos 10°) + cos 130°

$= 2\cos\frac{110°+10°}{2}.\cos\frac{110°-10°}{2} + \cos 130°$

$= 2.\frac{1}{2}.\cos 50° - \cos 50° = \cos 50° - \cos 50°$

= 0

Illustration 55

(i) Show that cos 20° . cos 40° . cos 80° = 1 / 8.

(ii) Prove that sin 20° sin 40° sin 80° = $\frac{\sqrt{3}}{8}$

(i) L.H.S. = $\frac{1}{2}$. (2 cos 40° . cos 20°) . cos 80°

= $\frac{1}{2}$ {cos (40° + 20°) + cos (40° – 20°)} cos 80°

= $\frac{1}{2}$ {cos 60° + cos 20°} cos 80° = $\frac{1}{2}\left(\frac{1}{2}+\cos 20^\circ\right)$ cos 80°

= $\frac{1}{4}$ cos 80° + $\frac{1}{2}$ cos 20° . cos 80°

= $\frac{1}{4}$ cos 80° + $\frac{1}{4}$ .2.cos 80° . cos 20°

= $\frac{1}{4}$ cos 80° + $\frac{1}{4}$ {cos (80° + 20°) + cos (80° – 20°)}

= $\frac{1}{4}$ cos (180° – 100°) + $\frac{1}{4}$ {cos 100° + cos 60°}

= $-\frac{1}{4}$ cos 100° + $\frac{1}{4}$ cos 100° + $\frac{1}{4}.\frac{1}{2}=\frac{1}{8}$

(ii) L.H.S. = sin 20° sin 40° sin 80° = $\frac{1}{2}$ (2 sin 80° sin 40°) sin 20°

= $\frac{1}{2}$ (cos 40° – cos 120°) sin 20° = $\frac{1}{4}$ (2 cos 40° sin 20° – 2 cos 120° sin 20°)

= $\frac{1}{4}\left[\sin(40^\circ+20^\circ)-\sin(40^\circ-20^\circ)-2\left(-\frac{1}{2}\right)\sin 20^\circ\right]$

= $\frac{1}{4}$ [sin 60° – sin 20° + sin 20°] = $\frac{1}{4}$ sin 60°

= $\frac{1}{4}.\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{8}$

Illustration 56

Prove that tan 20°. tan 40°. tan 80° = $\sqrt{3}$.

Solution L.H.S. $= \frac{\sin 20°}{\cos 20°}.\frac{\sin 40°}{\cos 40°}.\frac{\sin 80°}{\cos 80°}$

$$= \frac{(2\sin 40°.\sin 20°)\,.\,\sin 80°}{(2\cos 40°\,.\,\cos 20°).\,\cos 80°} = \frac{\{\cos(40°-20°)-\cos(40°+20°)\}.\,\sin 80°}{\{\cos(40°+20°)+\cos(40°-20°)\}.\,\cos 80°}$$

$$= \frac{(\cos 20°-\cos 60°)\,.\,\sin 80°}{(\cos 60°+\cos 20°)\,.\,\cos 80°} = \frac{\sin 80°\,.\,\cos 20° - \frac{1}{2}\sin 80°}{\frac{1}{2}\cos 80° + \cos 80°.\cos 20°}$$

$$= \frac{2\sin 80°\cos 20° - \sin 80°}{\cos 80° + 2\cos 80°\,.\,\cos 20°} = \frac{\{\sin(80°+20°)+\sin(80°-20°)\} - \sin 80°}{\cos 80° + \{\cos(80°+20°)+\cos(80°-20°)\}}$$

$$= \frac{\sin 100° + \sin 60° - \sin(180°-100°)}{\sin(180°-100°)+\cos 100°+\cos 60°} = \frac{\sin 100° + \frac{\sqrt{3}}{2} - \sin 100°}{-\cos 100° + \cos 100° + \frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Illustration 57

Prove that cos(30^0 – A) cos(30^0 + A) + cos(45^0 + A)cos(45^0 –A) = cos 2A + $\frac{1}{4}$

L.H.S. = $\frac{1}{2}$ [2 cos(30^0 – A) cos(30^0 + A) + 2 cos (45^0 + A) cos(45^0 – A)]

= $\frac{1}{2}$ [cos60^0 + cos 2A + cos 90^0 + cos 2A] = cos 2A + $\frac{1}{4}$

Illustration 58

Prove that sin A. sin (60° – A). sin (60° + A) = $\frac{1}{4}$ sin 3A

L.H.S. = sin A. sin (60° – A). sin (60° + A)

= $\frac{1}{2}$ sin A [2 sin (60° + A). sin (60° – A)]

= $\frac{1}{2}$ sin A [cos(60° + A – 60° + A) – cos(60° + A + 60° – A)]

= $\frac{1}{2}$ sin A (cos 2A – cos 120°)

$= \frac{1}{4}$ (2 cos 2A sin A – 2 cos 120° sin A)

$= \frac{1}{4}\left[\sin(2A + A) - \sin(2A - A) - 2\left(-\frac{1}{2}\right)\sin A\right]$

$= \frac{1}{4}$ (sin 3A – sin A + sin A) = $\frac{1}{4}$ sin 3A

Illustration 59

Show that sin A. sin (B – C) + sin B. sin (C – A) + sin C. sin (A – B) = 0

sin A. sin (B – C) + sin B. sin (C – A) + sin C. sin (A – B)

$= \frac{1}{2}$ [2 sin A. sin (B – C) + 2 sin B. sin (C – A) + 2 sin C. sin (A – B)]

$= \frac{1}{2}$ [cos (A – B + C) – cos (A + B – C) + cos (B – C + A) – cos (B + C – A) + cos (C – A + B) – cos (A + C – B)]

= 0

Illustration 60

If cosec A + sec A = cosec B + sec B, prove that tan $\frac{1}{2}$ (A + B) = cot A . cot B. We have

cosec A + sec A = cosec B + sec B

or, sec A – sec B = cosec B – cosec A, or, $\frac{1}{\cos A} - \frac{1}{\cos B} = \frac{1}{\sin B} - \frac{1}{\sin A}$

or, $\frac{\cos B - \cos A}{\cos A.\cos B} = \frac{\sin A - \sin B}{\sin A.\sin B}$, or, $\frac{\cos B - \cos A}{\sin A - \sin B} = \frac{\cos A.\cos B}{\sin A.\sin B}$

or, $\frac{2.\sin\frac{B+A}{2}.\sin\frac{A-B}{2}}{2.\cos\frac{A+B}{2}.\sin\frac{A-B}{2}}$ = cot A . cot B

or, $\tan\frac{A+B}{2}$ = cot A . cot B. i.e. $\tan\frac{1}{2}$ (A + B) = cot A . cot B.

Illustration 61

If sin θ = n sin (θ + 2α), prove that tan (θ + α) = $\frac{1+n}{1-n}$ tan α

Given, if sin θ = n sin (θ + 2α) or $\frac{1}{n} = \frac{\sin(\theta + 2\alpha)}{\sin\theta}$

or $\frac{1+n}{1-n} = \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta}$ [by componendo and dividendo]

$= \frac{2\sin\frac{\theta+2\alpha+\theta}{2}\cos\frac{\theta+2\alpha-\theta}{2}}{2\cos\frac{\theta+2\alpha+\theta}{2}\sin\frac{\theta+2\alpha-\theta}{2}} = \frac{2\sin(\theta+\alpha)\cos\alpha}{2\cos(\theta+\alpha)\sin\alpha}$

= tan (θ + α). cot α

or $\tan(\theta + \alpha) = \frac{1+n}{1-n}\tan\alpha$

Illustration 62

If $\sin\alpha = k\sin\beta$, prove that $\tan\frac{\alpha-\beta}{2} = \frac{k-1}{k+1}\tan\frac{\alpha+\beta}{2}$.

We have $\sin\alpha = k\sin\beta$, or, $\frac{\sin\alpha}{\sin\beta} = \frac{k}{1}$.

By componendo and dividendo, we get

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{k+1}{k-1} \quad \text{or,} \quad \frac{2\sin\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}} = \frac{k+1}{k-1}.$$

or, $\tan\frac{\alpha+\beta}{2}.\cot\frac{\alpha-\beta}{2} = \frac{k+1}{k-1}$

or, $\frac{k-1}{k+1}.\tan\frac{\alpha+\beta}{2} = \frac{1}{\cot\frac{\alpha-\beta}{2}} = \tan\frac{\alpha-\beta}{2}$.

Illustration 63

If $\cos(\theta - \alpha) = p$ and $\sin(\theta + \beta) = q$, prove that $p^2 + q^2 - 2pq\sin(\alpha + \beta) = \cos^2(\alpha + \beta)$.

Let $\alpha - \theta = A$ and $\theta + \beta = B$; then $A + B = \alpha + \beta$,

$p = \cos(-A) = \cos A$ and $q = \sin B$

L.H.S. $= \cos^2 A + \sin^2 B - 2\cos A.\sin B.\sin(A + B)$

$= \cos^2 A + \sin^2 B - \{\sin(A + B) - \sin(A - B)\}.\sin(A + B)$

$= \cos^2 A + \sin^2 B - \sin^2(A + B) + \sin(A + B).\sin(A - B)$

$= \cos^2 A + \sin^2 B - \sin^2(A + B) + \sin^2 A - \sin^2 B$

$= 1 - \sin^2(A + B) = \cos^2(A + B) = \cos^2(\alpha + \beta)$.

Illustration 64

If $\cos(A + B + C) = \cos A\cos B\cos C$, then prove that

$8\sin(B + C)\sin(C + A)\sin(A + B) + \sin 2A\sin 2B\sin 2C = 0$

$\cos A\cos(B + C) - \sin A\sin(B + C) = \cos A\cos B\cos C$

$\cos A[\cos(B + C) - \cos B\cos C] = \sin A\sin(B + C)$

or $\cos A(-\sin B\sin C) = \sin A\sin(B + C)$

$\sin(B + C) = -\frac{\cos A\sin B\sin C}{\sin A}$

Write similar values of $\sin(C + A)$, $\sin(A + B)$ and multiply and put

$\sin A\cos A = 1/2\sin 2A$.

Practice Assignment– V

1. Prove that

(i) $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$ (ii) $\frac{\sin 5x - \sin 3x}{\cos 5x + \cos 3x} = \tan x$

(iii) $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$ (iv) $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x+y}{2}\right)$

(v) $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$

2. Prove that

(i) $\sin 38^0 + \sin 22^0 = \sin 82^0$

(ii) $\sin 40^0 + \sin 20^0 = \cos 10^0$

(iii) $\sin 5\pi/18 - \cos 4\pi/9 = \sqrt{3} \sin \pi/9$

3. Prove that

(i) $\cos 55^0 + \cos 65^0 + \cos 175^0 = 0$

(ii) $\cos 20^0 + \cos 100^0 + \cos 140^0 = 0$

4. Prove that

(i) $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

(ii) $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(iii) $\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$

(iv) $\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

(v) $\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

(vi) $\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

5. Prove that

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$

(ii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos A/2 \cos 3A/2 \sin 3A$

6. Prove that

(i) $\sin 50^0 - \sin 70^0 + \sin 10^0 = 0$

(ii) $\sin 10^0 + \sin 20^0 + \sin 40^0 + \sin 50^0 - \sin 70^0 - \sin 80^0 = 0$

7. If cos A + cos B = 1/2 , sin A + sin B = 1/4 , prove that $\tan\left(\frac{A+B}{2}\right) = \frac{1}{2}$

8. Prove that $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2\cot^n \frac{A-B}{2}$ or, 0 according as n is even or odd positive integer.

9. Prove that

(i) $\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

(ii) $\cos 20^0 \cos 100^0 + \cos 100^0 \cos 140^0 - \cos 140^0 \cos 200^0 = -3/4$

(iii) $\sin \theta/2 \sin 7\theta/2 + \sin 3\theta/2 \sin 11\theta/2 = \sin 2\theta \sin 5\theta$

10. If cosec A + secA = cosec B + sec B , then prove that tan A tan B = $\cot\left(\frac{A+B}{2}\right)$.

11. If sin 2A = λ sin2B, prove that $\frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$.

12. If $\cos\alpha = \lambda\cos(\alpha - 2\beta)$, prove that $\tan(\alpha - \beta)\tan\beta = \frac{1-\lambda}{1+\lambda}$.

13. If $\sin\alpha = k\sin\beta$, prove that $\tan\frac{\alpha-\beta}{2} = \frac{k-1}{k+1}\tan\left(\frac{\alpha+\beta}{2}\right)$.

14. (i) If $\cos(A + B)\sin(C - D) = \cos(A - B)\sin(C + D)$, then show that $\tan A\tan B\tan C + \tan D = 0$.

(ii) If $\cos(A + B)\sin(C + D) = \cos(A - B)\sin(C - D)$, then show that $\cot A\cot B\cot C = \cot D$.

15. Prove that

(i) $\cos^2(A - B) + \cos^2 B - 2\cos A\cos B\cos(A - B) = \sin^2 A$

(ii) $\sin^2 A + \sin^2(A - B) - 2\sin A\cos B\sin(A - B) = \sin^2 B$

(iii) $\cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A + B) = \sin^2(A + B)$

16. If three angles A, B, C are in AP, prove that $\cot B = \frac{\sin A - \sin C}{\cos C - \cos A}$.

17. Prove that $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = 3$ if $\cot\alpha = 2\tan\beta$.

Multiple and Submultiple Angles

An angle of the form nA, where n is an integer is called a multiple angle, for example 2A, 3A, 4A, … etc. are multiple angles of A.

Here we shall express trigonometric ratios of multiple angles of A in terms of trigonometric ratios of A.

(a) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$, $\left[\theta \neq n\pi + \frac{\pi}{2}\right]$

(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$, $\left[\theta \neq n\pi + \frac{\pi}{2}\right]$

(c) $1 + \cos 2\theta = 2\cos^2\theta$, $1 - \cos 2\theta = 2\sin^2\theta$ or $\frac{1+\cos 2\theta}{2} = \cos^2\theta$, $\frac{1-\cos 2\theta}{2} = \sin^2\theta$

(d) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta'}$ where $\theta \neq (2n+1)\frac{\pi}{4}$, $\neq (2n + 1)\frac{\pi}{2}$, $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$, $\theta \neq \frac{n\pi}{2}$, $\theta \neq n\pi$

(e) $\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$, where, $\theta \neq n\pi$

(f) $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$, where $\theta \neq n\pi$

(g) $\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$, where $\theta \neq (2n+1)\pi$

(h) $\frac{1+\cos\theta}{1-\cos\theta} = \cot^2\frac{\theta}{2}$, where $\theta \neq 2n\pi$

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

(j) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$

(k) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$, $\theta \neq (2n+1)\frac{\pi}{2}$, $(2n+1)\frac{\pi}{6}$

(l) $\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$, $\theta \neq n\pi$, $\frac{n\pi}{3}$

(m) $\sin\theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$

(n) $\cos\theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$

(o) $\tan\theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$

Illustration 65

If $\tan\theta = \frac{b}{a}$, then find the value of $a\cos 2\theta + b\sin 2\theta$.

$a\cos 2\theta + b\sin 2\theta$

$$= a\frac{1-\tan^2\theta}{1+\tan^2\theta} + b\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \frac{a\left(1-\frac{b^2}{a^2}\right)}{1+\frac{b^2}{a^2}} + \frac{b.2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{\frac{a(a^2-b^2)}{a^2}}{\frac{a^2+b^2}{a^2}} + \frac{\frac{2b^2}{a}}{\frac{a^2+b^2}{a^2}}$$

$$= \frac{a(a^2-b^2)}{a^2+b^2} + \frac{2b^2.a}{a^2+b^2} = \frac{a(a^2-b^2+2b^2)}{a^2+b^2}$$

$$= \frac{a(a^2+b^2)}{a^2+b^2} = a$$

Illustration 66

Prove that $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$

$$\text{LHS} = \frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta} = \frac{2\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

Illustration 67

Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

$$\text{L.H.S.} = \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$$

$$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}}$$

$$= \sqrt{2+2\cos 2\theta}$$

$$= \sqrt{2(2\cos^2\theta)} = 2\cos\theta$$

Illustration 68

Prove that, tan α. tan(60° – α) tan(60° + α) = tan 3α.

$$\text{L.H.S} = \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin(60^\circ-\alpha)}{\cos(60^\circ-\alpha)}\cdot\frac{\sin(60^\circ+\alpha)}{\cos(60^\circ+\alpha)}$$

$$= \frac{\sin\alpha(\sin^2 60^\circ-\sin^2\alpha)}{\cos\alpha(\cos^2 60^\circ-\sin^2\alpha)} = \frac{\sin\alpha\left(\frac{3}{4}-\sin^2\alpha\right)}{\cos\alpha\left(\frac{1}{4}-\sin^2\alpha\right)}$$

[∵ $\sin(A + B)\sin(A - B) = \sin^2A - \sin^2B$ and $\cos(A + B)\cos(A - B) = \cos^2A - \sin^2B$]

$$= \frac{\sin\alpha(3-4\sin^2\alpha)}{\cos\alpha(1-4\sin^2\alpha)} = \frac{3\sin\alpha-4\sin^3\alpha}{\cos\alpha[1-4(1-\cos^2\alpha)]}$$

$$= \frac{3\sin\alpha-4\sin^3\alpha}{\cos\alpha(4\cos^2\alpha-3)} = \frac{3\sin\alpha-4\sin^3\alpha}{4\cos^3\alpha-3\cos\alpha} = \frac{\sin 3\alpha}{\cos 3\alpha} = \tan 3\alpha$$

Illustration 69

Show that $\cos^2(A - 120^\circ) + \cos^2A + \cos^2(A + 120^\circ) = \frac{3}{2}$.

L.H.S.

$= \frac{1}{2}\{2\cos^2(A - 120^\circ) + 2\cos^2 A + 2\cos^2(A + 120^\circ)\}$

$= \frac{1}{2}\{1 + \cos 2(A - 120^\circ) + 1 + \cos 2A + 1 + \cos 2(A + 120^\circ)\}$

[∵ $2\cos^2\theta = 1 + \cos 2\theta$]

$= \frac{1}{2}\{3 + \cos 2A + \cos(2A + 240^\circ) + \cos(2A - 240^\circ)\}$

$= \frac{1}{2}\{3 + \cos 2A + 2 . \cos 2A . \cos 240^\circ\}$

$= \frac{1}{2}\{3 + \cos 2A + 2 . \cos 2A . (-\frac{1}{2})\}$

[∵ $\cos 240^\circ = \cos(180^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$]

$= \frac{1}{2}\{3 + \cos 2A - \cos 2A\} = \frac{3}{2}$.

Illustration 70

Prove that, $\sin^3\alpha + \sin^3\left(\frac{2\pi}{3}+\alpha\right)+\sin^3\left(\frac{4\pi}{3}+\alpha\right) = -\frac{3}{4}\sin 3\alpha$

∵ $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

∴ L.H.S. $= \frac{1}{4}[3\sin\alpha - \sin 3\alpha] + \frac{1}{4}\left[3\sin\left(\frac{2\pi}{3}+\alpha\right) - \sin(2\pi + 3\alpha)\right] + \frac{1}{4}\left[3\sin\left(\frac{4\pi}{3}+\alpha\right) - \sin(4\pi + 3\alpha)\right]$

$= \frac{1}{4}[3\sin\alpha - \sin 3\alpha] + \frac{1}{4}\left[3\sin\left(\frac{2\pi}{3}+\alpha\right)-\sin 3\alpha\right]+\frac{1}{4}\left[3\sin\left(\frac{4\pi}{3}+\alpha\right)-\sin 3\alpha\right]$

[$\because$ sin (2π + θ) = sin θ and sin (4π + θ) = sin θ]

$$= \frac{3}{4}\sin\alpha + \frac{3}{4}\left[\sin\left(\frac{2\pi}{3}+\alpha\right)+\sin\left(\frac{4\pi}{3}+\alpha\right)\right] - \frac{1}{4}(\sin 3\alpha + \sin 3\alpha + \sin 3\alpha)$$

$$= \frac{3}{4}\sin\alpha + \frac{3}{4}2\sin(\pi+\alpha)\cos\frac{\pi}{3} - \frac{3}{4}\sin 3\alpha$$ [Using the formula of sin C + sin D]

$$= \frac{3}{4}\sin\alpha + \frac{3}{4}.2.(-\sin\alpha).\frac{1}{2} - \frac{3}{4}\sin 3\alpha$$

$$= \frac{3}{4}\sin\alpha - \frac{3}{4}\sin\alpha - \frac{3}{4}\sin 3\alpha = -\frac{3}{4}\sin 3\alpha$$

Illustration 71

If sin x + sin y = a and cos x + cosy = b, show that sin (x + y) = $\frac{2ab}{a^2+b^2}$

We have 2ab = 2 (sin x + sin y) (cos x + cos y)
= 2(sin x cos x + sin y cos y + sin x cos y + cos x sin y)
= sin 2 x + sin 2 y + 2 sin (x + y)
= 2 sin (x + y) cos (x – y) + 2 sin (x + y)
= sin (x + y) [cos (x – y) + 1] ....(1)

and $a^2 + b^2 = (\sin x + \sin y)^2 + (\cos x + \cos y)^2$
$= \sin^2 x + \sin^2 y + 2 \sin x \sin y + \cos^2 x + \cos^2 y + 2 \cos x \cos y$
$= (\sin^2 x + \cos^2 x) + (\sin^2 y + \cos^2 y) + 2 (\cos x \cos y + \sin x \sin y)$
= 2 + 2 cos (x – y)
= 2 [1 + cos (x – y)] ...(2)

Dividing (1) by (2), we get

$$\frac{2ab}{a^2+b^2} = \sin(x+y)$$

Illustration 72

If α and β are acute angles and cos 2α = $\frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, show that tan α = $\sqrt{2}$ tan β.

We have $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3-\cos 2\beta}$, or, $\frac{1}{\cos 2\alpha} = \frac{3-\cos 2\beta}{3\cos 2\beta - 1}$

By componendo and dividendo, we get

$$\frac{1-\cos 2\alpha}{1+\cos 2\alpha} = \frac{3-\cos 2\beta - 3\cos 2\beta + 1}{3 - \cos 2\beta + 3\cos 2\beta - 1}$$

or, $\tan^2\alpha = \frac{4-4\cos 2\beta}{2+2\cos 2\beta} = \frac{4(1-\cos 2\beta)}{2(1+\cos 2\beta)} = 2.\tan^2\beta$

$\therefore$ $\tan\alpha = \sqrt{2}\,.\tan\beta$, since α, β are both acute angles.

Illustration 73

If $\tan^2\alpha = 1 + 2\tan^2\beta$, prove that cos 2β = 1 + 2 cos 2α.

We have $\tan^2\alpha = 1 + 2\tan^2\beta$

or, $\frac{1-\cos 2\alpha}{1+\cos 2\alpha} = 1 + \frac{2.(1-\cos 2\beta)}{1+\cos 2\beta}$ $\left[\because \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}\right]$

or, $\frac{1-\cos 2\alpha}{1+\cos 2\alpha} = \frac{1+\cos 2\beta + 2 - 2\cos 2\beta}{1+\cos 2\beta} = \frac{3-\cos 2\beta}{1+\cos 2\beta}$

or, $\dfrac{1+\cos 2\alpha}{1-\cos 2\alpha} = \dfrac{1+\cos 2\beta}{3-\cos 2\beta}$

By componendo and dividendo,

$\dfrac{2}{2\cos 2\alpha} = \dfrac{4}{-2+2\cos 2\beta}$ or, $\dfrac{1}{\cos 2\alpha} = \dfrac{2}{-1+\cos 2\beta}$

or, $-1 + \cos 2\beta = 2 \cos 2\alpha$, or, $\cos 2\beta = 1 + 2 \cos 2\alpha$.

Illustration 74

Prove that $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$

L.H.S. $= \dfrac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ . \cos 10^\circ}$

Now $\cos 10^\circ - \sqrt{3} \sin 10^\circ = 2\left(\dfrac{1}{2}.\cos 10^\circ - \dfrac{\sqrt{3}}{2}.\sin 10^\circ\right)$

$= 2 (\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ)$

$= 2 \sin (30^\circ - 10^\circ) = 2. \sin 20^\circ = 2 . \sin (2 . 10^\circ)$

$= 2 . (2 \sin 10^\circ . \cos 10^\circ) = 4 . \sin 10^\circ . \cos 10^\circ$.

Hence L.H.S. $= \dfrac{4\sin 10^\circ . \cos 10^\circ}{\sin 10^\circ . \cos 10^\circ} = 4$.

Illustration 75

Prove that $\cos^4\dfrac{\pi}{8} + \cos^4\dfrac{3\pi}{8} + \cos^4\dfrac{5\pi}{8} + \cos^4\dfrac{7\pi}{8} = \dfrac{3}{2}$.

L.H.S. $= \dfrac{1}{4}\left\{\left(2\cos^2\dfrac{\pi}{8}\right)^2 + \left(2\cos^2\dfrac{3\pi}{8}\right)^2 + \left(2\cos^2\dfrac{5\pi}{8}\right)^2 + \left(2\cos^2\dfrac{7\pi}{8}\right)^2\right\}$

$= \dfrac{1}{4}\left\{\left(1+\cos\dfrac{\pi}{4}\right)^2 + \left(1+\cos\dfrac{3\pi}{4}\right)^2 + \left(1+\cos\dfrac{5\pi}{4}\right)^2 + \left(1+\cos\dfrac{7\pi}{4}\right)^2\right\}$ $\left[\because 2\cos^2\dfrac{A}{2} = 1+\cos A\right]$

$= \dfrac{1}{4}\left\{\left(1+\cos\dfrac{\pi}{4}\right)^2 + \left(1-\cos\dfrac{\pi}{4}\right)^2 + \left(1-\cos\dfrac{\pi}{4}\right)^2 + \left(1+\cos\dfrac{\pi}{4}\right)^2\right\}$

$\left[\because \cos\dfrac{3\pi}{4} = \cos\left(\pi - \dfrac{\pi}{4}\right) = -\cos\dfrac{\pi}{4}, \cos\dfrac{5\pi}{4} = \cos\left(\pi + \dfrac{\pi}{4}\right) = -\cos\dfrac{\pi}{4}\right.$

$\left.\text{and } \cos\dfrac{7\pi}{4} = \cos\left(2\pi - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4}\right]$

$= \dfrac{1}{4}\left\{\left(1+\dfrac{1}{\sqrt{2}}\right)^2 + \left(1-\dfrac{1}{\sqrt{2}}\right)^2 + \left(1-\dfrac{1}{\sqrt{2}}\right)^2 + \left(1+\dfrac{1}{\sqrt{2}}\right)^2\right\}$ $= \dfrac{1}{4}\left\{4\left(1+\dfrac{1}{2}\right)\right\} = \dfrac{3}{2}$.

Illustration 76

If $180^\circ < \theta < 270^\circ$ and $\sin\theta = -\dfrac{4}{5}$, find the values of $\sin\dfrac{1}{2}\theta$ and $\cos\dfrac{1}{2}\theta$.

Since $180^\circ < \theta < 270^\circ$ i.e. θ lies in third quadrant, $\sin\theta$ and $\cos\theta$ are both negative.

$\therefore \sin\theta = -\dfrac{4}{5}$ and $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\dfrac{16}{25}} = -\dfrac{3}{5}$

Again $90^\circ < \dfrac{\theta}{2} < 135^\circ$ i.e. $\dfrac{\theta}{2}$ lies in the second quadrant.

$\therefore$ sin $\frac{\theta}{2}$ is positive and cos $\frac{\theta}{2}$ is negative

We have 2 $\sin^2\frac{\theta}{2}$ = 1 – cosθ = 1 – $\left(-\frac{3}{5}\right) = 1 + \frac{3}{5} = \frac{8}{5}$.

or, $\sin^2\frac{\theta}{2} = \frac{4}{5}$; $\therefore \sin\frac{\theta}{2} = \frac{2}{\sqrt{5}}$

Again 2 $\cos^2\frac{\theta}{2}$ = 1 + cos θ = 1 + $\left(-\frac{3}{5}\right) = \frac{2}{5}$

or, $\cos^2\frac{\theta}{2} = \frac{1}{5}$; $\therefore \cos\frac{\theta}{2} = -\frac{1}{\sqrt{5}}$.

Practice Assignment– VI

1. Prove the following identities :

(i) $\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}$

(ii) $\frac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\frac{\pi}{4}-\theta\right)$

(iii) 2 sin A $\cos^3$A – 2 $\sin^3$A cos A = (1/2) sin 4A

(iv) $\frac{1+\sin 2\theta - \cos 2\theta}{1+\sin 2\theta + \cos 2\theta} = \tan\theta$

(v) sec x = $\frac{2}{\sqrt{2+\sqrt{2+2\cos 4x}}}$

(vi) $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

2. Prove the following identities :

(i) tan(π/4 + θ) – tan(π/4 – θ) = 2 tan 2θ

(ii) tan A + cot A = 2 cosec 2A

(iii) (tan α $\sec^2$ α + cot α cosec^2α) – (tan α + cot α)
= 2 cosec 2α ($\sec^2$α + cosec^2α - 3)

(iv) cot A – tan A = 2 cot2A

(v) tan A + 2 tan2A + 4 tan4A + 8 cot 8A = cot A

(vi) cosec A – 2 cot 2A cos A = 2 sin A

(vii) cosec 2A – cot 2A = tan A

3. Prove

(i) sin 2x + 2 sin 4x + sin 6x = 4 $\cos^2$x sin 4x

(ii) tan 4x = $\frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$

(iii) cos 4x = 1 – 8 $\sin^2$ x $\cos^2$ x

(iv) cos 6x = 32 $\cos^6$ x – 48 $\cos^4$ x + 18 $\cos^2$ x – 1

4. Prove the following identities :

(i) 2 cos x – cos 3x – cos 5x = 16 $\cos^3$x $\sin^2$x

(ii) 1 + cos 2x + cos 4x + cos 6x = 4 cos x cos 2x cos 3x

5. Prove that

(i) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

(ii) $(\cos\beta - \cos\alpha)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

6. If $\tan\theta = \frac{a}{b}$, prove that $b\cos 2\theta + a\sin 2\theta = b$

7. If $2\cos\theta = x + \frac{1}{x}$, prove that $2\cos 3\theta = x^3 + \frac{1}{x^3}$

8. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then show that

(i) $\sin(\theta+\phi) = \frac{2ab}{a^2+b^2}$ (ii) $\cos(\theta+\phi) = \frac{b^2-a^2}{b^2+a^2}$

(iii) $\cos(\theta-\phi) = \frac{a^2+b^2-2}{2}$

9. If α, β be two distinct angles satisfying the equation $a\cos\theta + b\sin\theta = c$, show that

(i) $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$

(ii) $\tan(\alpha+\beta) = \frac{2ab}{a^2-b^2}$

10. Show that

(i) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

(ii) $\frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta} = \tan^2\frac{\theta}{2}$

11. Show that

(i) $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$

(ii) $\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\sin^2 2x$

12. Show that

$$\frac{1}{\cot\theta - \cot 3\theta} + \frac{1}{\tan 3\theta - \tan\theta} = \cot 2\theta$$

13. Prove that $\cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$, if $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$

14. Prove that :
(i) $\sqrt{3}\,\text{cosec}\,20^0 - \sec 20^0 = 4$
(ii) $4\cos 20^0 - \sqrt{3}\cot 20^0 = -1$

15. Prove that :
(i) $1 + \cos 56^0 + \cos 58^0 - \cos 66^0 = 4\cos 28^0\cos 29^0\sin 33^0$
(ii) $\cos^2 73^0 + \cos^2 47^0 + \cos 73^0\cos 47^0 = 3/4$
(iii) $\sin^2 12^0 + \sin^2 21^0 + \sin^2 39^0 + \sin^2 48^0 = 1 + \sin^2 9^0 + \sin^2 18^0$

16. Prove that :
(i) $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$
(ii) $\tan A\sec 4A + \tan 4A = \tan A + \tan 4A\sec 2A$

17. Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, $\tan\frac{x}{2}$ if

(i) tan x = $-\frac{4}{3}$, x in quadrant II

(ii) cos x = $-\frac{1}{3}$, x in quadrant III

(iii) sin x = $\frac{-1}{2}$, x in quadrant IV

Some Important Trigonometric Results

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$

$$= \sin\left(\frac{1\text{st}\angle + \text{last}\angle}{2}\right)\frac{\sin\frac{n\beta}{2}}{\sin\left(\frac{\beta}{2}\right)} = \frac{\sin\left(\alpha+(n-1)\frac{\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\beta/2}$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \cos\left(\frac{1\text{st}\angle + \text{last}\angle}{2}\right)\frac{\sin\frac{n\beta}{2}}{\sin\left(\frac{\beta}{2}\right)} = \frac{\cos\left(\alpha+(n-1)\frac{\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\beta/2}$$

(iii) $\cos A \cos 2A \cos 2^2 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

Illustration 77

Prove that $\cos\frac{\pi}{14} + \cos\frac{3\pi}{14} + \cos\frac{5\pi}{14} = \frac{1}{2}\cot\frac{\pi}{14}$

Here $\alpha = \frac{\pi}{14}$, $\beta = \frac{\pi}{14}$, and n = 3

$$S = \frac{\cos\left[\frac{\pi}{14} + \left(\frac{3-1}{2}\right)\frac{2\pi}{14}\right]\sin\left(\frac{2\pi}{14}\times\frac{3}{2}\right)}{\sin\left(\frac{2\pi}{14}\times\frac{1}{2}\right)}$$

$$S = \frac{\sin\frac{6\pi}{14}}{2\sin\frac{\pi}{14}} = \frac{\frac{1}{2}\sin\left(\frac{\pi}{2}-\frac{\pi}{14}\right)}{\sin\frac{\pi}{14}} = \frac{1}{2}\cot\frac{\pi}{14}$$

Illustration 78

Prove that $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$

Given expression can be written as $\frac{\sin\left(2^4\frac{2\pi}{15}\right)}{2^4\sin\frac{2\pi}{15}} = \frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}} = \frac{1}{16}$

Values Of T. Ratios Of Some Important Angles

(a) $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$

(b) $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$

(c) $\tan 15^0 = 2-\sqrt{3} = \cot 75^0$

(d) $\cot 15^0 = 2+\sqrt{3} = \tan 75^0$

(e) $\sin 22\frac{1}{2}^o = \frac{1}{2}\left(\sqrt{2-\sqrt{2}}\right) = \cos 67\frac{1}{2}^\circ$

(f) $\cos 22\frac{1}{2}^o = \frac{1}{2}\left(\sqrt{2+\sqrt{2}}\right) = \sin 67\frac{1}{2}^\circ$

(g) $\tan 22\frac{1}{2}^o = \sqrt{2}-1 = \cot 67\frac{1}{2}^\circ$

(h) $\cot 22\frac{1}{2}^o = \sqrt{2}+1 = \tan 67\frac{1}{2}^\circ$

(i) $\sin 18^o = \frac{\sqrt{5}-1}{4} = \cos 72^o$

(j) $\cos 18^o = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^o$

(k) $\sin 36^o = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^o$

(l) $\cos 36^o = \frac{\sqrt{5}+1}{4} = \sin 54^o$

(m) $\sin 9^o = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81^o$

(n) $\cos 9^o = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81^o$

Illustration 79

Prove that $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$.

$\frac{\pi}{5} = \frac{180^\circ}{5} = 36^\circ$ and $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$; $\therefore \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

$\therefore \quad \cos\frac{2\pi}{5} = 2\cos^2\frac{\pi}{5} - 1 = 2\left(\frac{\sqrt{5}+1}{4}\right)^2 - 1 = \frac{2(6+2\sqrt{5})}{16} - 1$

$$= \frac{3+\sqrt{5}}{4} - 1 = \frac{3+\sqrt{5}-4}{4} = \frac{\sqrt{5}-1}{4}$$

$$\therefore \quad \text{L.H.S.} = \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}+1-\sqrt{5}+1}{4} = \frac{2}{4} = \frac{1}{2}.$$

Illustration 80

Prove that tan 6° . tan 42° . tan 66° . tan 78° = 1.

$$\text{L.H.S.} = \frac{\sin 6^\circ . \sin 42^\circ . \sin 66^\circ . \sin 78^\circ}{\cos 6^\circ . \cos 42^\circ . \cos 66^\circ . \cos 78^\circ}$$

$$= \frac{(2\sin 66^\circ . \sin 6^\circ)(2\sin 78^\circ . \sin 42^\circ)}{(2\cos 66^\circ . \cos 6^\circ)(2\cos 78^\circ . \cos 42^\circ)}$$

$$= \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 120^\circ + \cos 36^\circ)}$$

$$= \frac{\left(\frac{1}{2} - \sin 18^\circ\right)\left(\cos 36^\circ + \frac{1}{2}\right)}{\left(\frac{1}{2} + \sin 18^\circ\right)\left(-\frac{1}{2} + \cos 36^\circ\right)} \qquad [\because \cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ]$$

$$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)}$$

$$= \frac{(3-\sqrt{5})(3+\sqrt{5})}{(1+\sqrt{5})(\sqrt{5}-1)}$$

$$= \frac{9-5}{5-1} = \frac{4}{4}$$

$$= 1$$

Expression of sin A/2 and cos A/2 in terms of sin A

We have $\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2 = 1 + \sin A$ and $\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 = 1 - \sin A$

so that $\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$ and $\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}$

By adding and subtracting, we have

$$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A} \qquad \text{....(i)}$$

and $$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A} \qquad \text{...(ii)}$$

In each of the formulae (i) and (ii) there are two ambiguous signs. To find these ambiguities we proceed as follows. We have

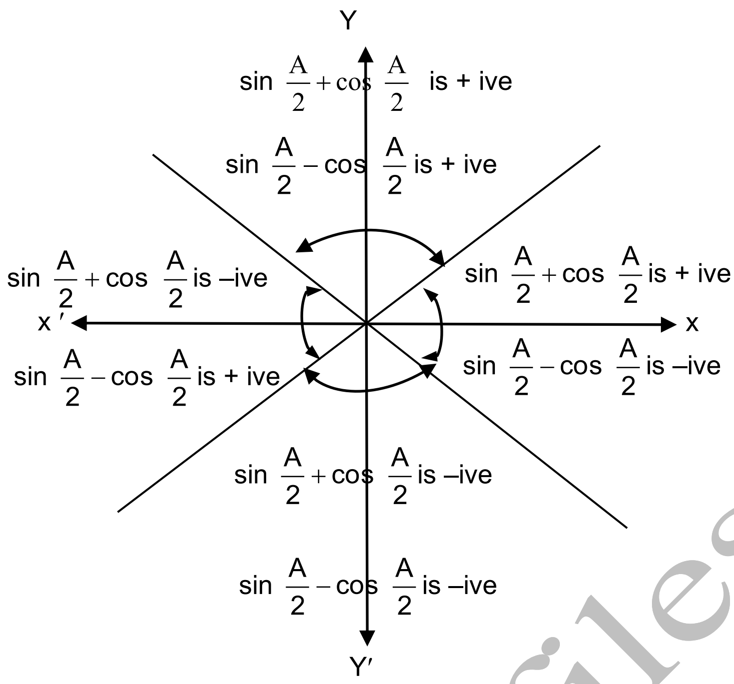
The RHS of this equation is positive if

$$2n\pi < \frac{\pi}{4} + \frac{A}{2} < (2n+1)\pi \qquad \text{i.e. if } 2n\pi - \frac{\pi}{4} < \frac{A}{2} < 2n\pi + \frac{3\pi}{4}$$

Hence $\sin\frac{A}{2} + \cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lies between $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$ and otherwise it is negative.

Similarly, $\sin\frac{A}{2} - \cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lies between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$ and otherwise it is negative.

These results can be shown graphically as given in the following figure.



If $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$. Then this is possible when

$$\sin\frac{A}{2} + \cos\frac{A}{2} = +\sqrt{1+\sin A}$$

and $\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A}$ (sin $\frac{A}{2}$ is +ve and numerically greater than cos $\frac{A}{2}$ when $\frac{A}{2}$ lies between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$)

Illustration 81

If A = 580^0 , then prove that sin A/2 = $\frac{1}{2}\left[-\sqrt{1+\sin A} - \sqrt{1-\sin A}\right]$

Since A/2 = 290^0, we have 225^0 < A /2 < 315^0

⇒ sin A/2 + cos A/2 and sin A/2 – cos A/2 are both negative

⇒ sin A/2 + cos A/2 = $-\sqrt{1+\sin A}$

⇒ sin A/2 – cos A/2 = $-\sqrt{1-\sin A}$

Adding these, 2 sin A/2 = $-\sqrt{1+\sin A} - \sqrt{1-\sin A}$

Illustration 82

Show that 4 cos 9° = $\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}$

We have $2\cos\frac{A}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$.

Putting A = 18°, we get $\frac{A}{2}$ = 9° and cos 9° > 0.

∴ $2\cos 9° = \sqrt{1+\sin 18°} + \sqrt{1-\sin 18°}$

$$= \sqrt{1+\frac{\sqrt{5}-1}{4}} + \sqrt{1-\frac{\sqrt{5}-1}{4}} = \sqrt{\frac{3+\sqrt{5}}{4}} + \sqrt{\frac{5-\sqrt{5}}{4}}$$

$$= \frac{\sqrt{3+\sqrt{5}}}{2} + \frac{\sqrt{5-\sqrt{5}}}{2}$$

Hence $4\cos 9^\circ = \sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}$.

Maximum and Minimum Values of Trigonometric Functions

As we know that $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$, $-\infty < \tan x < \infty$, $|\sec x| \ge 1$ and $|\text{cosec } x| \ge 1$.
If there is a trigonometric function of the form $a \sin x + b \cos x$,
then by putting $a = r\cos\theta$, $b = r\sin\theta$,
we have $a \sin x + b \cos x = r\cos\theta \sin x + r\sin\theta\cos x = r \sin(x+\theta)$,

where $r = \sqrt{a^2+b^2}$, $\tan\theta = \frac{b}{a}$

Since $-1 \le \sin(x+\theta) \le 1$ for all values of x. Therefore $-r \le r\sin(x+\theta) \le r$ for all x

$\Rightarrow \quad -\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$ for all x.

Hence the maximum and minimum values of a trigonometric function of the form asinx + bcosx are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

Illustration 83

Prove that $5\cos\theta + 3\cos(\theta + \pi/3) + 3$ lies between –4 and 10

The given expression is $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$$= \frac{1}{2}[13\cos\theta - 3\sqrt{3}\sin\theta] + 3 \qquad \ldots\ldots(1)$$

Let $r\cos\alpha = 13$, $r\sin\alpha = 3\sqrt{3}$

$\Rightarrow \quad r = 14$

Hence, (1) becomes $7\cos(\theta+\alpha) + 3$ which lies between –4 and 10.

Practice Assignment– VII

1. Prove that :
(i) $\cos 2\pi/7 + \cos 4\pi/7 + \cos 6\pi/7 = -1/2$
(ii) $\cos \pi/11 + \cos 3\pi/11 + \cos 5\pi/11 + \cos 7\pi/11 + \cos 9\pi/11 = 1/2$
(iii) $\cos 0 + \cos \pi/7 + \cos 2\pi/7 + \cos 3\pi/7 + \cos 4\pi/7 + \cos 5\pi/7 + \cos 6\pi/7 = 1$

2. Prove that :
(i) $\cos 2\pi/15 \cos 4\pi/15 \cos 8\pi/15 \cos 16\pi/15 = 1/16$
(ii) $\cos \pi/7 \cos 2\pi/7 \cos 4\pi/7 = -1/8$
(iii) $\cos \pi/33 \cos 2\pi/33 \cos 4\pi/33 \cos 8\pi/33 \cos 16\pi/33 = 1/32$
(iv) $\cos \pi/15 \cos 2\pi/15 \cos 3\pi/15 \cos 4\pi/15 \cos 5\pi/15 \cos 6\pi/15 \cos 7\pi/15 = 1/128$
(v) $\sin \pi/14 \sin 3\pi/14 \sin 5\pi/14 \sin 7\pi/14 \sin 9\pi/14 \sin 11\pi/14 \sin 13\pi/14 = 1/64$

3. If $\theta = \frac{\pi}{2^n+1}$, prove that $2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = 1$

4. Prove that $\cos^2 48^\circ - \sin^2 12^\circ = \frac{\sqrt{5}+1}{8}$

5. Prove that $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = -\frac{1}{4}$

6. Prove that $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=\frac{-1}{2}$

7. Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}=\frac{5}{16}$

8. Prove that $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}=\frac{1}{16}$

9. Prove that $\sin 12^o \sin 48^o \sin 54^o = \frac{1}{8}$

10. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$

11. Prove that $\tan 6^o \tan 42^o \tan 66^o \tan 78^o = 1$

12. Prove that $\cos 36^o \cos 42^o \cos 60^o \cos 78^o = \frac{1}{16}$

13. Prove that $\cos 6^0 \cos 42^0 \cos 66^0 \cos 78^0 = 1/16$

14. Prove that $\cot 7\frac{1}{2}^0 = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

15. Show that (a) $\cot 22\frac{1}{2}^0=\sqrt{2}+1$ (b) $\tan 11\frac{1}{4}^0=\sqrt{4+2\sqrt{2}}-\sqrt{2}-1$

16. Prove that $\tan 9^0 - \tan 27^0 - \tan 63^0 + \tan 81^0 = 4$

17. Prove that $4\sin 27^0 = (5+\sqrt{5})^{1/2}-(3-\sqrt{5})^{1/2}$

18. Show that the maximum and minimum values of $8\cos\theta - 15\sin\theta$ lies between -17 and 17.

19. Find the maximum and minimum value of
(i) $\sin x + \cos x$
(ii) $7\cos x + 24\sin x$
(iii) $3\cos x + 5\sin\left(x-\frac{\pi}{6}\right)$

20. Find the interval in which A lies if
(i) $A = \sin^2\theta + \cos^4\theta$
(ii) $A = \sin^8\theta + \cos^{14}\theta$

Conditional Trigonometric Identities

We have certain trigonometric identities like, $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$ etc. Such identities are identities in the sense that they hold for all values of the angles, which satisfy the given condition among them and they are called conditional trigonometric identities. If A, B, C denotes the angles of a triangle then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratios.

(i) If $A + B + C = \pi$, then $A + B = \pi - C$, $B + C = \pi - A$, $C + A = \pi - B$

(ii) If $A + B + C = \pi$, then $\sin(A + B) = \sin C$, $\sin(B + C) = \sin A$, $\sin(C + A) = \sin B$

(iii) If $A + B + C = \pi$, then $\cos(A + B) = -\cos C$, $\cos(B + C) = -\cos A$ and $\cos(C + A) = -\cos B$

(iv) If $A + B + C = \pi$, $\sin\left(\frac{A+B}{2}\right)=\cos\frac{C}{2}$, $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$, $\tan\left(\frac{A+B}{2}\right)=\cot\frac{C}{2}$

All problems on conditional identities are broadly divided into the following types :

(i) Identities involving sines and cosines of the multiple or sub-multiples of the angles involved.

(ii) Identities involving squares of the sines and cosines of the multiples or sub-multiples of the angles involved.

(iii) Identities involving tangents and cotangents of the multiples or sub-multiples of the angles involved.

Type-1. Identities involving sines and cosines of the multiple or submultiple of the angles involved. Working method :

1. Express the first two terms as product using C and D formulae
2. In the product obtained in step 1 replace the sum of two angles in terms of the third by using given relation
3. Expand the third term by using formulae (Double angle change into single angle or change into half angle)
4. Taking common factor.
5. Express the trigonometric ratio of the single angle in terms of the remaining angles
6. Use the one of the formulae given in the step 1 to convert the sum into product.

Illustration 84

If A + B + C = π , prove that cos A + cos B+ cos C = 1 + 4 sin $\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$.

LHS = cos A + cos B + cos C

$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$

$\Rightarrow \quad 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)-2\sin^2\left(\frac{C}{2}\right)+1$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]+1$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]+1$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]+1$

$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]+1$

$\Rightarrow \quad 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)=\text{RHS}$

Illustration 85

If A + B + C = π, prove that cos 2A + cos 2B – cos 2C = 1 – 4 sin A sin B cos C.

L.H.S. = (cos 2 A + cos 2B) – cos 2C

$= 2\cos\frac{2A+2B}{2}\cos\frac{2A-2B}{2}-\cos 2C$

= 2 cos (A + B) cos (A – B) – (2 $\cos^2$ C – 1)

= 2 cos (π – C) cos (A – B) – 2 $\cos^2$ C + 1

= 1 – 2 cos C cos (A – B) – 2 $\cos^2$ C

= 1 – 2 cos C [cos (A – B) + cos C]

= 1 – 2 cos C[cos (A – B) + cos (π – (A + B))]

= 1 – 2 cos C[cos (A – B) – cos (A + B)]

$= 1-2\cos C\left[2\sin\frac{2A}{2}\sin\frac{2B}{2}\right]$

= 1 – 2 cos C [2 sin A sin B] = 1 – 4 sin A sin B cos C = R.H.S.

Type-2. Identities involving squares of sines and cosines of multiple or sub-multiples of the angles involved.

Working method :

(i) Arrange the terms on the LHS of the identity so that either

$\sin^2A - \sin^2B = \sin(A+B)\sin(A-B)$

or $\cos^2A - \sin^2B = \cos(A-B)\cos(A+B)$ can be used.

(ii) Take the common factor outside.

(iii) Express the trigonometric ratio of a single angle inside the bracket into that of the sum of the angles.

(iv) Use the formulas to convert the sum into product.

Illustration 86

If A + B + C = π, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.

LHS $= \cos^2 A + \cos^2 B + \cos^2 C$

$\Rightarrow \cos^2 A + (1 - \sin^2 B) + \cos^2 C$

$\Rightarrow (\cos^2 A - \sin^2 B) + \cos^2 C + 1$ [because $A + B = \pi - C$, $\cos(A + B) = -\cos C$]

$\Rightarrow \cos(A + B)\cos(A - B) + \cos^2 C + 1$

$\Rightarrow -\cos C\cos(A - B) + \cos^2 C + 1$

$\Rightarrow -\cos C[\cos(A - B) - \cos C] + 1$

$\Rightarrow -\cos C[\cos(A - B) + \cos(A + B)]$ [because $\cos C = -\cos(A + B)$]

$\Rightarrow -\cos C[2\cos A\cos B] + 1$

$\Rightarrow 1 - 2\cos A\cos B\cos C$ = RHS

Illustration 87

If A + B + C = π, prove that $\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A\cos B\sin C$.

L.H.S. $= (\sin^2 A - \sin^2 B) + \sin^2 C$

$= \sin(A + B)\sin(A - B) + \sin^2 C$

$= \sin(\pi - C)\sin(A - B) + \sin^2 C$ $(\because A + B + C = \pi)$

$= \sin C\sin(A - B) + \sin^2 C$

$= \sin C[\sin(A - B) + \sin C]$

$= \sin C[\sin(A - B) + \sin(\pi - (A + B))]$

$= \sin C[\sin(A - B) + \sin(A + B)]$

$$= \sin C\left[2\sin\left(\frac{2A}{2}\right)\cos\left(\frac{-2B}{2}\right)\right]$$

$= \sin C[2\sin A\cos B] = 2\sin A\cos B\sin C$ = R.H.S.

Illustration 88

If A + B + C = π, prove that $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

We have $2\sin^2\frac{A}{2} = 1 - \cos A$, or, $\sin^2\frac{A}{2} = \frac{1}{2}(1 - \cos A)$

Similarly, $\sin^2\frac{B}{2} = \frac{1}{2}(1 - \cos B)$.

L.H.S. $= \frac{1}{2}(1 - \cos A) + \frac{1}{2}(1 - \cos B) + \sin^2\frac{C}{2}$

$= \frac{1}{2}\{2 - (\cos A + \cos B)\} + \sin^2\frac{C}{2}$

$= 1 - \frac{1}{2}.2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \sin^2\frac{C}{2}$

$= 1 - \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \sin^2\frac{C}{2}$

$$\left[\because A + B + C = \pi; \therefore \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}\right]$$

$= 1 - \sin\frac{C}{2}\cos\frac{A-B}{2} + \sin^2\frac{C}{2}$

$= 1 - \sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$

$= 1 - \sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$

$$\left[\because A+B+C=\pi; \because \frac{C}{2}=\frac{\pi}{2}-\frac{A+B}{2}\right]$$

$$= 1 - \sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]$$

$$= 1 - \sin\frac{C}{2}\left[2\sin\frac{A}{2}.\sin\frac{B}{2}\right]$$

$$= 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Illustration 89

If A + B + C = π, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$.

We have $2\cos^2\frac{A}{2} = 1+\cos A$, or $\cos^2\frac{A}{2}=\frac{1}{2}(1+\cos A)$

Similarly, $\cos^2\frac{B}{2}=\frac{1}{2}(1+\cos B)$

L.H.S. $= \frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)+\cos^2\frac{C}{2}$.

$$= \frac{1}{2}\{2+(\cos A+\cos B)+\cos^2\frac{C}{2}$$

$$= 1+\frac{1}{2}.2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos^2\frac{C}{2}$$

$$= 1+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}+\cos^2\frac{C}{2} \qquad \left[\because A+B+C=\pi; \therefore \frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}\right]$$

$$= 1+\sin\frac{C}{2}\cos\frac{A-B}{2}+1-\sin^2\frac{C}{2} \qquad = 2+\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]$$

$$= 2+\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right] \left[\because A+B+C=\pi; \therefore \frac{C}{2}=\frac{\pi}{2}-\frac{A+B}{2}\right]$$

$$= 2+\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]$$

$$= 2+\sin\frac{C}{2}\left(2.\sin\frac{A}{2}\sin\frac{B}{2}\right) \qquad = 2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

Illustration 90

If A + B + C = π, prove that $\cos^2 A - \cos^2 B - \cos^2 C = 2\cos A \sin B \sin C - 1$.

We have $2\cos^2 A = 1+\cos 2A$, or, $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

Similarly, $\cos^2 B = \frac{1}{2}(1+\cos 2B)$

L.H.S. $= \frac{1}{2}(1+\cos 2A)-\frac{1}{2}(1+\cos 2B)-\cos^2 C$

$$= \frac{1}{2}(\cos 2A-\cos 2B)-\cos^2 C$$

$= \frac{1}{2} . 2. \sin(A+B) . \sin(B-A) - \cos^2 C$

$= \sin(\pi - C)\sin(B-A) - (1-\sin^2 C)$ [$\therefore$ A + B + C = π]

$= \sin C \sin(B-A) + \sin^2 C - 1$

$= \sin C[\sin(B-A) + \sin C] - 1$

$= \sin C[-\sin(A-B) + \sin\{\pi - (A+B)\}] - 1$ [$\therefore$ A + B + C = π]

$= \sin C[\sin(A+B) - \sin(A-B)] - 1$ [$\therefore \sin\{\pi - (A+B)\} = \sin A + B)$]

$= \sin C. [2\cos A \sin B] - 1$

$= 2\cos A \sin B \sin C - 1.$

Type – 3. Identities for tangent and cotangent of the angles

(i) Express the sum of two angles in terms of third angle by using given relation
(ii) Taking tan from both sides
(iii) Expand the LHS in step 2 by using formula for the tangent of the compound angles.
(iv) Use cross multiplication in expression obtained in step 3
(v) Arrange the terms as per the requirement in the sum.

Illustration 91

If A, B, C are the angles of a triangle, then prove that
tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C.

We have A + B + C = π

$\Rightarrow$ $2A + 2B = 2\pi - 2C$

$\therefore$ $\tan(2A + 2B) = \tan(2\pi - 2C)$

$\Rightarrow$ $\frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan 2B} = -\tan 2C$

$\Rightarrow$ $\tan 2A + \tan 2B = -\tan 2C + \tan 2A \tan 2B \tan 2C$

$\Rightarrow$ $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

Illustration 92

If A, B, C be the angles of a triangle, then prove that
$\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1.$

Since A, B, C, are the angles of a triangle, $\therefore$ A + B + C = π, or, A + B = π – C;

$\therefore \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$

or, $\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}.\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$,

or, $\tan\frac{A}{2}.\tan\frac{C}{2} + \tan\frac{B}{2}.\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$

or, $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1.$

Illustration 93

If x + y + z = xyz, then prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$

Let x = tan A, y = tan B, z = tan C

⇒ tan A + tan B + tan C = tan A. tan B . tan C
⇒ A + B + C = nπ
⇒ 2A + 2B + 2C = 2nπ
⇒ tan(2A + 2B + 2C) = 0
⇒ $\frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C}{1 - \tan 2A \tan 2B - \tan 2B \tan 2C - \tan 2C \tan 2A} = 0$
⇒ tan 2A + tan 2B + tan 2C = tan 2A. tan 2B. tan 2C

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Conditional Identities To Remember

If A + B + C = 180°, then
(i) sin 2A + sin 2B + sin 2C = 4sin A sin B sin C.
(ii) cos 2A + cos 2B + cos 2C = –1 – 4 cos A cos B Cos C
(iii) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
(v) tan A + tan B + tan C = tan A tan B tan C
(vi) $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$
(vii) cot A cot B + cot B cot C + cot C cot A = 1
(viii) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

Practice Assignment– VIII

If A + B + C = π, prove the following identities :
1. sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C
2. cos 2A + cos 2B + cos 2C = – 1 – 4 cos A cos B cos C
3. sin A + sin B + sin C = 4 cosA/2 cos B/2 cosC/2
4. cos A + cos B + cos C = 1 + 4 sin A/2 sin B/2 sin C/2
5. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
6. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$
7. cos A/2 + cos B/2 + cos C/2 = 4 cos{(B + C)/4} cos{(C + A)/4} cos{(A + B)/4}
8. tan A + tan B + tan C = tan A tan B tan C
9. cot A cot B + cot B cot C + cot C cot A = 1
10. cot (A/2) + cot (B/2) + cot (C/2) = cot (A/2) cot (B/2) cot (C/2)
11. $\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1} = 8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

12. $$\frac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}=\frac{\tan(A/2)}{\tan(C/2)}$$

13. $$\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

14. If x + y + z = xyz then prove that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\,\frac{2y}{1-y^2}\,\frac{2z}{1-z^2}$$

15. If xy + yz +zx = 1, then prove that

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}=\frac{2}{\left[(1+x^2)(1+y^2)(1+z^2)\right]^{1/2}}$$

Objective Assignment

1. If $\tan\theta = a - \frac{1}{4a}$, then $\sec\theta - \tan\theta$ is equal to

(a) $-2a, \frac{1}{2a}$ **(b)** $-\frac{1}{2a}, 2a$

(c) 2a **(d)** $\frac{1}{2a}, 2a$

2. $\text{Sec}^2\theta = \frac{4xy}{(x+y)^2}$, where $x\in R$, $y\in R$, is true iff

(a) $x + y \neq 0$ **(b)** $x = y, x \neq 0$

(c) $x = y$ **(d)** $x \neq 0, y \neq 0$

3. $\sin^2\theta = \frac{(x+y)^2}{4xy}$, $x \in R$, $y\in R$ gives real θ if and only if

(a) $x + y = 0$ **(b)** $x = y$

(c) $|x| = |y| \neq 0$ **(d)** None of these

4. If $\sin\theta + \text{cosec}\,\theta = 2$, then the value of $\sin^8\theta + \text{cosec}^8\theta$ is =

(a) 2 **(b)** 2^8

(c) 2^4 **(d)** None of these

5. If $x = r\sin\theta.\cos\phi$, $y = r\sin\theta.\sin\phi$ and $z = r\cos\theta$ then the value of $x^2 + y^2 + z^2$ is independent of

(a) θ, ϕ **(b)** r, θ

(c) r, ϕ **(d)** r

6. Let $f(\theta) = \frac{\cot\theta}{1+\cot\theta}$ & $\alpha + \beta = \frac{5\pi}{4}$. Then the value of $f(\alpha).f(\beta)$ is

(a) 2 **(b)** $-\frac{1}{2}$

(c) $\frac{1}{2}$ **(d)** none of these

7. If $0 < \beta < \alpha < \frac{\pi}{4}$, $\cos(\alpha+\beta) = \frac{3}{5}$ and $\cos(\alpha-\beta) = \frac{4}{5}$, then $\sin 2\alpha$ is equal to

(a) 1 **(b)** 0

(c) 2 **(d)** none of these

8. If $\cos\alpha = \frac{1}{2}\left(x+\frac{1}{x}\right)$, $\cos\beta = \frac{1}{2}\left(y+\frac{1}{y}\right)$, then $\cos(\alpha-\beta)$ is equal to

(a) $\frac{x}{y}+\frac{y}{x}$ **(b)** $xy+\frac{1}{xy}$

(c) $\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$ **(d)** none of these

9. If $\frac{2\sin\alpha}{1+\sin\alpha+\cos\alpha}=\lambda$, then $\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha}$ is equal to

(a) $\frac{1}{\lambda}$ **(b)** λ

(c) $1-\lambda$ **(d)** $1+\lambda$

10. If a sec α – c tan α = d, b sec α + d tan α = c, then

(a) $a^2+c^2=b^2+d^2$ **(b)** $a^2+d^2=b^2+c^2$

(c) $a^2+b^2=c^2+d^2$ **(d)** ab = cd

11. If α, β, γ & δ be four angles of a cyclic quadrilateral, then the value of $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta$ is

(a) 1 **(b)** 0

(c) –1 **(d)** none of these

12. If $0^o<\theta^o<180^o$ then $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots.+\sqrt{2(1+\cos\theta)}}}}$ there being n no. of 2′s, is equal to

(a) $2\cos\frac{\theta}{2^n}$ **(b)** $2\cos\frac{\theta}{2^{n-1}}$

(c) $2\cos\frac{\theta}{2^{n+1}}$ **(d)** none of these

13. The value of $\sqrt{3}$ cosec 20^o – sec 20^o is equal to

(a) 2 **(b)** 4

(c) $2\,\frac{\sin 20^o}{\sin 40^o}$ **(d)** $4\,\frac{\sin 20^o}{\sin 40^o}$

14. The value of cosec10^o – $\sqrt{3}$ sec 10^o is equal to

(a) $\frac{1}{2}$ **(b)** 2

(c) 4 **(d)** 8

15. For all real values of θ, cot θ – 2 cot 2 θ is equal to

(a) tan 2 θ **(b)** tan θ

(c) – cot 3 θ **(d)** none of these

16. The value of tan $\frac{\pi}{16}$ + 2tan $\frac{\pi}{8}$ + 4 is equal to

(a) cot $\frac{\pi}{8}$ **(b)** cot $\frac{\pi}{16}$

(c) $\cot\frac{\pi}{16}-4$ **(d)** None of these

17. If tan $\frac{\pi}{9}$, x and tan $\frac{5\pi}{18}$ are in A.P. and tan $\frac{\pi}{9}$, y and tan $\frac{7\pi}{18}$ are also in A.P. then

(a) 2x = y **(b)** x > y

(c) x = y **(d)** None of these

18. If cos (x – y), cos x and cos(x + y) are in H.P. then |cos x secy/2 | equals

(a) 1 **(b)** 2

(c) $\sqrt{2}$ **(d)** none of these

19. If 2 sin α. cos β sin γ = sin β sin ($\alpha + \gamma$) then tan α, tan β and tan γ are in
(a) A.P. **(b)** G.P.
(c) H.P. **(d)** None of these

20. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of the equation $8x^2 - 26x + 15 = 0$ then $\cos(\alpha + \beta)$ is equal to
(a) $\frac{-627}{725}$ **(b)** $\frac{627}{725}$
(c) –1 **(d)** none of these

21. If sin α + sin β = a, cos α – cosβ = b, then $\tan\frac{\alpha-\beta}{2}$ is equal to
(a) $-\frac{a}{b}$ **(b)** $-\frac{b}{a}$
(c) $\sqrt{a^2 + b^2}$ **(d)** none of these

22. Let a = cos A + cos B – cos (A + B), b = 4 sin $\frac{A}{2}$ sin $\frac{B}{2}$ cos $\frac{A+B}{2}$. Then a – b is equal to
(a) 1 **(b)** 0
(c) –1 **(d)** none of these

23. If cos 20° – sin 20° = P, then cos 40° is equal to
(a) $-P\sqrt{2-P^2}$ **(b)** $P\sqrt{2-P^2}$
(c) $P + \sqrt{2-P^2}$ **(d)** none of these

24. If 3 sin θ + 4 cos θ = 5, then the value of 4 sin θ – 3 cos θ is
(a) 0 **(b)** 5
(c) 1 **(d)** none of these

25. If cos 2x + 2 cos x = 1 then $\sin^2 x\,(2 - \cos^2 x)$ is equal to
(a) 1 **(b)** –1
(c) $-\sqrt{5}$ **(d)** $\sqrt{5}$

26. The value of tan 63° – cot 63° is equal to
(a) $\frac{2}{\sqrt{5}+1}.\sqrt{10-2\sqrt{5}}$ **(b)** $\frac{2}{\sqrt{5}+1}.\sqrt{10+2\sqrt{5}}$
(c) $\frac{\sqrt{5}-1}{4}.\sqrt{10-2\sqrt{5}}$ **(d)** none of these

27. The value of cos 9° – sin 9° is
(a) $-\sqrt{\frac{5-\sqrt{5}}{2}}$ **(b)** $\frac{5+\sqrt{5}}{4}$
(c) $\frac{1}{2}\sqrt{5-\sqrt{5}}$ **(d)** none of these

28. The value of $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$ is
(a) 0 **(b)** $\sqrt{5}$
(c) 1 **(d)** none of these

29. The value of tan 20° + 2tan50° – tan70° is
(a) 1 **(b)** 0
(c) tan 50° **(d)** none of these

30. If | tan A | < 1, and | A | is acute then $\frac{\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}}$ is equal to
(a) tan A **(b)** –tan A
(c) cot A **(d)** – cot A

31. The set of all possible values of α in $[-\pi, \pi]$ s.t $\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}$ is equal to $\sec\alpha - \tan\alpha$ is

(a) $\left[0, \frac{\pi}{2}\right)$
(b) $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$
(c) $[-\pi, 0]$
(d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

32. $\tan\theta \,.\, \tan\left(\frac{\pi}{3}+\theta\right) \tan\left(\frac{\pi}{3}-\theta\right)$ is equal to

(a) $\tan 2\theta$
(b) $\tan 3\theta$
(c) $\tan^3\theta$
(d) $-\cot\theta$

33. If $\tan\theta + \tan\left(\theta+\frac{\pi}{3}\right) + \tan\left(\theta+\frac{2\pi}{3}\right) = k\tan 3\theta$, then k is equal to

(a) 1
(b) 3
(c) $\frac{1}{3}$
(d) none of these

34. The value of $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}.\sin\frac{7\pi}{14}.\sin\frac{9\pi}{14}.\sin\frac{11\pi}{14}.\sin\frac{13\pi}{14}$ is equal to

(a) 1
(b) $\frac{1}{16}$
(c) $\frac{1}{64}$
(d) none of these

35. The numerical value of $\sin\frac{\pi}{18}.\sin\frac{5\pi}{18}.\sin\frac{7\pi}{18}$ is equal to

(a) 1
(b) $\frac{1}{8}$
(c) $\frac{1}{4}$
(d) $\frac{1}{2}$

36. The value of $\cos 12^\circ.\cos 24^\circ.\cos 36^\circ.\cos 48^\circ.\cos 72^\circ.\cos 84^\circ$ is

(a) $\frac{1}{64}$
(b) $\frac{1}{32}$
(c) $\frac{1}{16}$
(d) $\frac{1}{128}$

37. The value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$ is

(a) 0
(b) 1
(c) $\frac{1}{2}$
(d) none of these

38. $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}$ is equal to

(a) $\frac{n}{2}$
(b) $\frac{n-1}{2}$
(c) $\frac{n}{2}-1$
(d) none of these

39. The value of $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\ldots$ to n terms is equal to

(a) 1
(b) 0

(c) $\frac{n}{2}$ **(d)** none of these

40. The value of sin 78° – sin 66° – sin 42° + sin 6° is

(a) $\frac{1}{2}$ **(b)** $-\frac{1}{2}$

(c) – 1 **(d)** none of these

41. If $4n\alpha = \pi$ then $\cot\alpha.\cot 2\alpha .\cot 3\alpha.... \cot (2n-1)\alpha$ is equal to

(a) 1 **(b)** –1

(c) ∞ **(d)** none of these

42. Let p = a cos θ - b sin θ. Then for all real θ

(a) $p > \sqrt{a^2 + b^2}$ **(b)** $p < -\sqrt{a^2+b^2}$

(c) $-\sqrt{a^2+b^2} \le p \le \sqrt{a^2+b^2}$ **(d)** none of these

43. The maximum value of $1+\sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ for real values of θ is

(a) 3 **(b)** 5

(c) 4 **(d)** None of these

44. The minimum value of cos 2θ + cos θ for real values of θ is

(a) $-\frac{9}{8}$ **(b)** 0

(c) – 2 **(d)** none of these

45. The least value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$ is

(a) $4 + \sqrt{10}$ **(b)** $4 - \sqrt{10}$

(c) 0 **(d)** None of these

ANSWERS

PRACTICE ASSIGNMENT I

1. (i) 324° (ii) –229°5′27″ (app) (iii) 210° (iv) 39° 22′ 30″ (v) 300°
2. (i) $\frac{5\pi}{3}$ (ii) $\frac{251}{360}\pi$ (iii) $\frac{\pi}{24}$ (iv) $\frac{5\pi}{36}$ (v) $\frac{-19}{72}\pi$ (vi) $\frac{4\pi}{3}$ (vii) $\frac{26\pi}{9}$
3. 45°, $\frac{5\pi}{12}$
4. 12π
5. 12°36′
6. $\frac{20\pi}{3}$
7. 5 : 4
8. (i) $\frac{2}{15}$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$
9. $\frac{20\pi}{3}$
10. 38° 30′
11. 0.204 miles
12. 2.217 m
13. 65° 27′ 16″
14. 8 and 4

PRACTICE ASSIGNMENT II

12. $\sec\theta = \frac{17}{8}$, $\tan\theta = \frac{15}{8}$

PRACTICE ASSIGNMENT III

1. (i) $\frac{1}{2}$ (ii) $-\sqrt{2}$ (iii) $\frac{-1}{\sqrt{3}}$ (iv) $\frac{-1}{\sqrt{2}}$ (v) $\frac{\sqrt{3}}{2}$ (vi) $-\frac{\sqrt{3}}{2}$

   (vii) –1 (viii) –2 (ix) 2 (x) $\frac{1}{\sqrt{3}}$ (xi) 2 (xii) –2

2. 16/3
3. $\frac{-1}{16}$
7. (i) 1 (ii) 1
11. (i) $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\cot\theta = \frac{4}{3}$, $\sec\theta = \frac{5}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$

    (ii) $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\cot\theta = \frac{-1}{\sqrt{3}}$, $\sec\theta = -2$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$

    (iii) $\sin\theta = -\frac{4}{5}$, $\operatorname{cosec}\theta = -\frac{5}{4}$, $\cos\theta = -\frac{3}{5}$, $\sec\theta = -\frac{5}{3}$, $\tan\theta = \frac{4}{3}$

    (iv) $\sin\theta = -\frac{12}{13}$, $\operatorname{cosec}\theta = -\frac{13}{12}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = -\frac{12}{5}$, $\cot\theta = -\frac{5}{12}$

    (v) $\sin\theta = \frac{5}{13}$, $\operatorname{cosec}\theta = \frac{13}{5}$, $\cos\theta = -\frac{12}{13}$, $\sec\theta = -\frac{13}{12}$, $\cot\theta = -\frac{12}{5}$

12. $\frac{-7}{2}$
13. $\frac{3}{5}$
14. 2
15. 2
19. 0
20. $x = \sin\theta$

PRACTICE ASSIGNMENT IV

1. (i) $\frac{-56}{65}$ (ii) $\frac{-33}{65}$ (iii) $\frac{56}{33}$ (iv) $\frac{-16}{65}$ (v) $\frac{-63}{65}$ (vi) $\frac{16}{33}$
2. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (iii) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (iv) $\sqrt{3} - 2$

PRACTICE ASSIGNMENT VI

17. (i) $\sqrt{5}/5$, $2\sqrt{5}/5$, 2 (ii) $\sqrt{6}/3$, $-\sqrt{3}/3$, $-\sqrt{2}$ (iii) $\frac{\sqrt{2-\sqrt{3}}}{2}$, $\frac{-\sqrt{2+\sqrt{3}}}{2}$, $-\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$

PRACTICE ASSIGNMENT VII

19. (i) $\sqrt{2}$ and $-\sqrt{2}$ (ii) 25 and –25 (iii) $\sqrt{19}$ and $-\sqrt{19}$

20. (i) $\frac{3}{4} \le A \le 1$ (ii) $0 < A \le 1$

OBJECTIVE ASSIGNMENT

1	A	11	B	21	B	31	D	41	A
2	B	12	A	22	A	32	B	42	C
3	C	13	B	23	B	33	B	43	C
4	A	14	C	24	A	34	C	44	A
5	A	15	B	25	A	35	B	45	B
6	C	16	B	26	A	36	A		
7	A	17	A	27	C	37	C		
8	C	18	C	28	A	38	C		
9	B	19	C	29	B	39	B		
10	C	20	A	30	C	40	B		

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